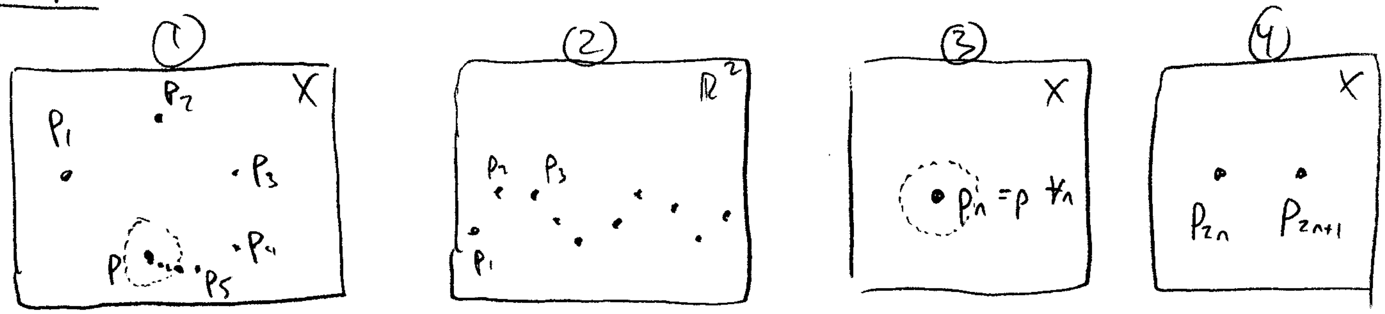
MTHSC 453

Lecture 15 Convergence sequences.

<u>Recall</u>: A sequence $\{p_n\}$ in X is a function $f: \mathbb{N} \to X$, mapping $n \longmapsto p_n$.

<u>Question</u>: What does it mean for a sequence to <u>converge</u>?

<u>Examples</u>:



Converges? Yes No Yes No.

But <u>why</u>?

<u>Def</u>: A sequence $\{p_n\}$ <u>converges</u> if $\exists\, p \in X$ s.t.

$$\boxed{\forall \varepsilon > 0,\ \exists N \text{ s.t. } n \geq N \Rightarrow d(p_n, p) < \varepsilon.}$$

<u>Write</u>: $p_n \to p$ or $\lim_{n \to \infty} p_n = p$.

Say "p_n converges to p" or "p is the limit of the sequence $\{p_n\}$"

Ex: $p_n = \frac{n+1}{n}$ in $\mathbb{R}$.

Claim: $p_n \to 1$. [Goal: Given $\varepsilon > 0$, how to find n s.t. $\left|\frac{n+1}{n} - 1\right| < \varepsilon$?
$\Rightarrow \frac{1}{n} < \varepsilon$]

Proof: Given $\varepsilon > 0$, choose $N = \lceil \frac{1}{\varepsilon} \rceil + 1$.

For if $n \geq N$, then $n \geq \frac{1}{\varepsilon}$, hence $\frac{1}{n} \leq \varepsilon$.

So $|p_n - p| = \left| \frac{n+1}{n} - 1 \right| = \left| \frac{1}{n} \right| < \varepsilon$. $\square$

True or False?

(A) If $p_n \to p$ & $p_n \to q$ $\Rightarrow p = q$?

(B) If $\{p_n\}$ bounded $\Rightarrow$ p_n converges?

(C) If p_n converges $\Rightarrow$ $\{p_n\}$ bounded?

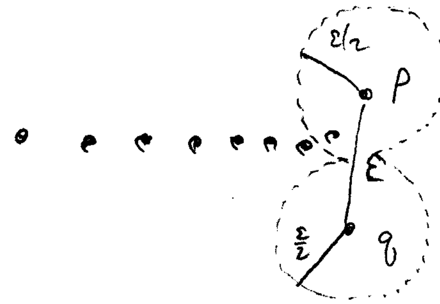
(D) $p_n \to p$ $\Rightarrow$ p is a l.p. of range $\{p_n\}$?

(E) p is a l.p. of $E \subset X$ $\Rightarrow$ $\exists$ seq. $\{p_n\}$ in E s.t. $p_n \to p$?

(F) $p_n \to p$ $\Leftrightarrow$ Every nbhd of p contains all but finitely many points from $\{p_n\}$?

Answers:

(A) True.



Assume $p_n \to p$, $p_n \to q$, $p \neq q$.

Let $\varepsilon = d(p,q) > 0$.

Then $\exists\, N_p$ s.t. $n \geq N_p$ implies $d(p,q) < \frac{\varepsilon}{2}$.

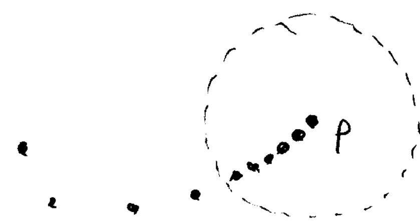
Also, $\exists\, N_q$ s.t. $n \geq N_q$ implies $d(p,q) < \frac{\varepsilon}{2}$.

Let $\boxed{N = \max\{N_p, N_q\}}$.

Then $n \geq N$ implies $\varepsilon < d(p,q) < d(p,p_n) + d(p_n,q) < \varepsilon$. ↯

(B) False. (See Ex (4) above).

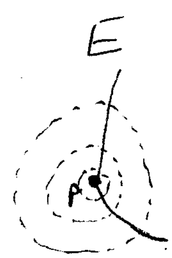
(C) True. Use $\varepsilon > 1$, so $\exists\, N$ s.t. $n \geq N$



$\Rightarrow d(p_n,p) < 1$.

Let $R = \max\{1, d(p_k,p) : k = 1, \dots, N\}$.

So all $p_n \in B_R(p)$. ✓



(D) False (See Ex (3) above.)

(E) True: Choose $p_n \in B_{1/n}(p)$

(F) True. (Exercise.)