MTHSC 453

Lecture 16 Subsequences, Cauchy sequences.

Theorem: Let $\{s_n\}$, $\{t_n\}$ be sequences in $\mathbb{C}$, and $s_n \to s$, $t_n \to t$.

Then $\lim_{n\to\infty} (s_n + t_n) = s + t$.

Proof: (Idea: Bound $|(s_n+t_n) - (s+t)| \leq |s_n - s| + |t_n - t| < \varepsilon$.)

Given $\varepsilon > 0$, $\exists N_s, N_t$ s.t. $n > N_s \Rightarrow |s_n - s| < \frac{\varepsilon}{2}$

$n > N_t \Rightarrow |t_n - t| < \frac{\varepsilon}{2}$.

Let $N = \max\{N_s, N_t\}$.

Then, for $n > N$, $|(s_n+t_n) - (s+t)| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$,

as desired. Thus $s_n + t_n \to s + t$. $\square$

Theorem: If $s_n \to s$, then $\lim_{n\to\infty} c s_n = cs$, and $\lim_{n\to\infty} (c + s_n) = c + s$.

(Idea: $|cs_n - cs| = |c| \cdot |s_n - s|$.)

Proof: Fix $\varepsilon > 0$. Then $\exists N$ s.t. $n \geq N \Rightarrow |s_n - s| < \frac{\varepsilon}{|c|}$.

Then for this N, $n \geq N \Rightarrow |cs_n - cs| = |c| \cdot |s_n - s| \leq |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon$. ✓

Proof of $\lim_{n\to\infty} (c + s_n) = c + s$ left as exercise. $\square$

Theorem: If $s_n \to s$, $t_n \to t$, then $\lim_{n\to\infty} s_n t_n = st$.

(Idea: $|s_n t_n - st| = |(s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)|$.)

Proof: Given $\varepsilon > 0$, let $K = \max\{|s|, |t|, 1, \varepsilon\}$ ↖ "later" (not apparent right away.)

$\exists\ N_1, N_2$ s.t. $n \geq N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{3K}$,

$n \geq N_2 \Rightarrow |t_n - t| < \frac{\varepsilon}{3K}$

Let $N = \max\{N_1, N_2\}$.

Then $|s_n t_n - st| \leq |(s_n - s)(t_n - t)| + |s(t_n - t)| + |t(s_n - s)|$

$$< \frac{\varepsilon^2}{9K^2} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \quad \text{if } K \geq 1$$

$$\text{if } K \geq \varepsilon \quad < \frac{\varepsilon}{9K} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon. \quad \square$$

Subsequences:

Let $\{p_n\}$ be a sequence. Let $n_1 < n_2 < n_3 < \cdots$ in $\mathbb{N}$ be an increasing sequence.

Then $\{p_{n_i}\}$ is a subsequence.

Ex: $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\}$

$n_1 = 2 \quad n_2 = 4 \quad n_3 = 5$

Question: Does this sequence converge?

Does the subsequence converge?

Prop: If $p_n \to p$, then any subsequence converges to p.

Proof (sketch). Every nbhd of p contains all but finitely many pts of $\{p_n\}$, and hence finitely many pts of $\{p_{n_i}\}$.

Ex: The sequence $\{1, \pi, \frac{1}{2}, \pi, \frac{1}{3}, \pi, \frac{1}{4}, \pi, \dots\}$ does not converge, but has convergence subsequences (to subsequential limits).

Question: Must every sequence contain a conv. subsequence?

Ans: No. Ex: $\{1, 2, 3, \dots\}$.

Does every bounded sequence contain a conv. subsequence?

Ans: No: In $\mathbb{Q}$, $\{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \dots\}$.

But when does a bounded sequence contain a conv. subsequence.

Def: A metric space is sequentially compact if every sequence has a convergent subsequence.

Theorem: X compact $\Longleftrightarrow$ X is sequentially compact.

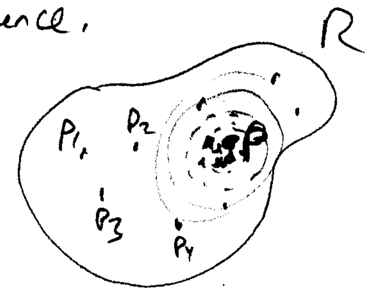
("$\Rightarrow$" says "In a cpt space, every sequence has a convergent subsequence.")

Cor (Bolzano-Weierstrass): In $\mathbb{R}^k$, every bounded sequence has a convergent subsequence.

Proof (of theorem).

($\Rightarrow$) Let $R = \text{range}\{p_n\}$.

- If R is finite, then some p in $\{p_n\}$ is acheived infinitely many times; use this subsequence.



- If R is infinite, then by previous thm, since X is compact, R has a limit point, called p.

  Use this to construct subsequence. (Use $N_{1/n}(p)$ for $n=1,2,\ldots$) ✓

($\Leftarrow$) Much harder. We'll skip (not in Rudin) ✓ $\square$

Cauchy Sequences

Motivation: How to tell if $\{p_n\}$ converges if we don't know its limit?

Idea: If they do converge, then the p_n must be getting close to each other.

Def: $\{p_n\}$ is a <u>Cauchy sequence</u> means that:

$$\boxed{\forall \varepsilon > 0,\ \exists N \text{ s.t. } m, n \geq N \Rightarrow d(p_m, p_n) < \varepsilon.}$$

Theorem: If $\{p_n\}$ converges, then $\{p_n\}$ is Cauchy.

Proof: [Idea: Want to bound $d(p_n, p_m) \leq \underbrace{d(p_n, p) + d(p, p_m)}_{\text{we know these are small.}}$]

Given $\varepsilon > 0$, $\exists N$ s.t. $n \geq N \Rightarrow d(p, p_n) < \varepsilon/2$.

So for <u>this N</u>, $n, m \geq N \Rightarrow d(p_n, p_m) \leq d(p_n, p) + d(p, p_m)$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

as desired. ✓ □

Question: Are Cauchy sequences convergent?

Ans: No! Ex: $\{3, 3.1, 3.14, 3.141, 3.1415, \ldots\}$.

But it some sense it "should" be true.

Def: A metric space X is <u>complete</u> if every Cauchy sequence converges to a point of X.

Ex: $\mathbb{Q}$ is <u>not</u> complete

$\mathbb{R}$ is complete (we'll see why later)