MTHSC 453

Lecture 17 Complete spaces

Recall: $\{x_n\}$ converges $\Longrightarrow$ $\{x_n\}$ is Cauchy.

We say that a m.s. X is complete if every Cauchy sequence converges. (i.e, if $\Longleftarrow$ holds.)

Theorem: Compact metric spaces are complete.

[e.g., $X = [a,b]$, or X = finite set, etc]

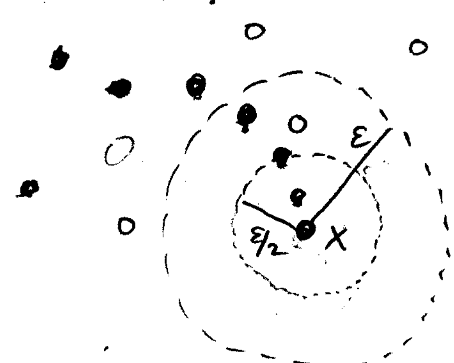
Proof: Let $\{x_n\}$ be Cauchy in X.

Since X is compact $\Rightarrow$ X is sequentially compact

every seq. has a conv. subseq.

So $\exists$ subsequence $\{x_{n_k}\}$ converging to some $x \in X$.

Claim: $\{x_n\} \longrightarrow x$.



Fix $\varepsilon > 0$. $\{x_n\}$ Cauchy means

$\exists N_1$ s.t. $i,j \geq N \Rightarrow d(x_i, x_j) < \varepsilon/2$.

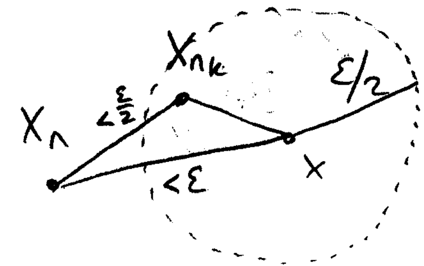
Also, $\{x_{n_k}\} \longrightarrow x$ implies $\exists N_2$ s.t. $n_k > N_2 \Rightarrow d(x_{n_k}, x) < \varepsilon/2$.

Let $N = \max\{N_1, N_2\}$.

If $n \geq N$, then $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x)$ for any fixed $n_k > N$.

$$< \quad \varepsilon/2 \quad + \quad \varepsilon/2 = \varepsilon.$$

So, given $\varepsilon > 0$, we found N that shows $x_n \to x$.



Since $\{x_n\}$ was arbitrary, X is complete. $\square$

Cor: $[a,b]$ is complete. [Or any k-cell in $\mathbb{R}^k$.] $\square$

Cor: $\mathbb{R}^n$ is complete.

Proof: If $\{x_n\}$ is Cauchy, it's bounded (exercise), so it's contained in some (compact) k-cell. So $\{x_n\}$ converges. $\square$

Non-examples • $(0,1]$ is not complete.

e.g., $\{\frac{1}{n}\}$ does not converge in $(0,1]$.

• $\mathbb{Q}$ is not complete.

e.g., $\{3, 3.1, 3.14, 3.141, 3.14159, 3.141592, \ldots\}$ does not converge in $\mathbb{Q}$.

Ex: Does $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}$ converge?

assume $n > m$

Consider $|X_n - X_m| = \left(\frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n}\right) \geq \frac{n-m}{n} = 1 - \frac{m}{n}$.

Let $n = 2m$, $|X_{2n} - X_n| > \frac{1}{2}$, so seq. is not Cauchy,

so it doesn't converge.

Ex: $X_1 = 1$, $X_2 = 2$, define recursively $X_n = \frac{1}{2}(X_{n-2} + X_{n-1})$.

Easy to show that this sequence is Cauchy.

So it converges (though we have no idea where it converges to!)

Question: If X is not complete, can it be embedded in one that is? (e.g., $\mathbb{Q} \subset \mathbb{R}$).

Answer: Yes!

Theorem: Every metric space (X, d) has a completion (X^*, d).

Examples • The completion of $\mathbb{Q}$ is $\mathbb{R}$.

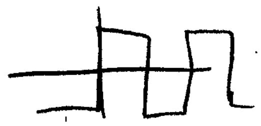
• Consider the space X of all polynomials over $\mathbb{R}$, with

$d(f, g) = \int_a^b |f - g| \, dx$.

Functions like $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$, $\cos x$, $\sin x$, are not in this space, but they are in the completion of X

(4)

Consider the space (generated by) finite sums of $\sin nx$ & $\cos nx$.

The square wave  is not in this space, but

it's in the <u>completion</u>, because $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$.

<u>Question</u>: Given X, how do we construct X^*?

<u>Idea</u>: let $X^* = \left\{ \begin{array}{l} \text{all Cauchy sequences in } X \text{ under} \\ \text{the equiv. relation } \sim \end{array} \right\}$

where $\boxed{\begin{array}{l} \{p_n\} \sim \{q_n\} \quad (\text{"same"}) \\ \quad \text{if } \lim_{n\to\infty} d(p_n, q_n) \to 0. \end{array}}$

For $P, Q \in X^*$ define <u>distance</u> as

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n),$$ where $\{p_n\}$ and $\{q_n\}$ are

representatives for P & Q.

HW!

* Need to show this defn is <u>well-defined</u>. (and that lim. exists!)

<u>Claim</u>: X^* is <u>complete</u>, with X <u>isometrically embedded</u> in X^*

[i.e, $\exists$ bijection from X to a <u>subset</u> of X^*; dist. preserving]

In fact, for $p \in X$, $p \longmapsto [(p, p, p, p, \ldots)]$

equiv. class containing this sequence.

<u>Remark</u>: This is an <u>alternative</u> way to construct $\mathbb{R}$!

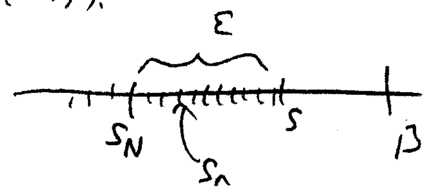
Bounded sequences

Def: A sequence $\{s_n\}$ is monotonically increasing if $s_{n+1} \geq s_n \ \forall n$

" decreasing if $s_{n+1} \leq s_n \ \forall n$.

Theorem: Bounded monot. sequences converge [to their sup or inf]

Proof. Given $\{s_n\}$, monot. incr, let $s = \sup(\text{range}\{s_n\})$.



So $\forall \varepsilon > 0$, $\exists N$ s.t. $s - \varepsilon < s_N$.

But then $\forall n \geq N$, $s_n \geq s_N \Rightarrow s - \varepsilon < s_N < s_n$

$\Rightarrow s_n \to s$. ✓

The case where $\{s_n\}$ is monot. decr. is analogous (exercise.) ✓ □

Write $\boxed{s_n \to +\infty}$ if $\forall M \in \mathbb{R}$, $\exists N$ s.t. $n \geq N \Rightarrow s_n > M$.

Similarly, write $s_n \to -\infty$ ....

- Given $\{s_n\}$, let $E = \{\text{subsequential limits}\}$ (allow $\pm\infty$).

This set is closed (see Rudin).

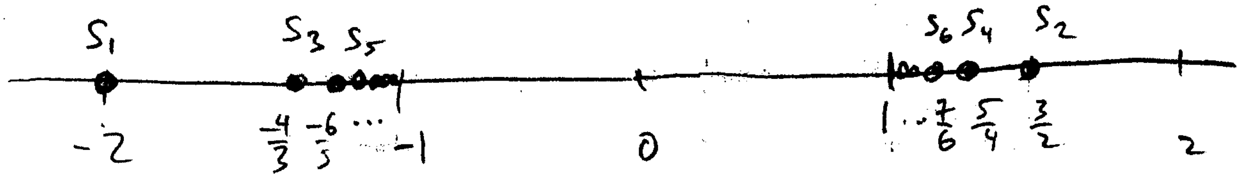
Let $s^* = \sup E \leftarrow \limsup s_n$, or "upper limit" of E.

$s_* = \inf E \leftarrow \liminf s_n$, or "lower limit" of E.

Alternate:

$$\boxed{\begin{aligned} \limsup s_n &= \lim_{n\to\infty}\left(\sup_{k \geq n} s_k\right) \\ \liminf s_n &= \lim_{n\to\infty}\left(\inf_{k \geq n} s_k\right). \end{aligned}}$$

<u>Example</u>: Let $s_n = (1+\frac{1}{n})(-1)^n$

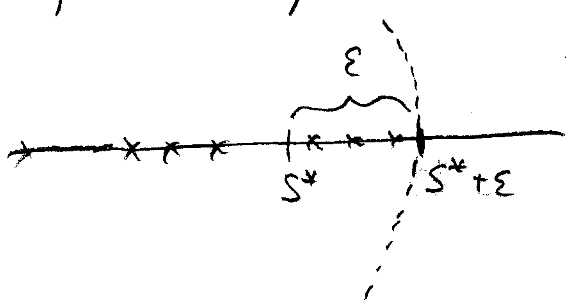


Note that $E = \{-1, 1\}$, $\liminf s_n = -1$

$\limsup s_n = 1$

<u>Remark</u>: If $x > s^*$, that does <u>not</u> imply that $x > s_n$ for all n, but it <u>does</u> imply that $x > s_n$ <u>for all but finitely many</u> n, or "eventually, $x > s_n$."

Thus, $\forall \varepsilon > 0$, $\exists N$ s.t. $n \geq N$, $s_n - s^* < \varepsilon$



(or $s_n - \varepsilon < s^*$)