# MTHSc 453

## $\boxed{\text{Lecture 18}}$ Series

Question: What does this mean? $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{\pi^2}{6}$.

Or this: $1 - 1 + 1 - 1 +/- \dots$

$$\overset{?}{=} (1-1) + (1-1) + \dots = 0$$

or $\overset{?}{=} 1 + (-1+1) + (-1+1) + \dots = 1$

Euler: This should $= \boxed{\frac{1}{2}}$!

Some background:

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots = \frac{1}{1 - 1/3}$$ "geometric series"

In general: $1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$. But why??

Maybe: $(1 + x + x^2 + x^3 + \dots)(1-x) = 1$ Does this make sense?

For all $x$? What if $x = 2$?

For all $x$? Try $\boxed{x=2}$: $1 + 2 + 4 + 8 + \dots = \frac{1}{1-2} = -1$

Try $\boxed{x=-2}$: $1 - 2 + 4 - 8 +/- \dots = \frac{1}{1-(-2)} = \frac{1}{3}$

Try $\boxed{x=-1}$: $1 - 1 + 1 - 1 +/- \dots = \frac{1}{1-(-1)} = \frac{1}{2}$

According to Euler!

Is this meaningful?

- We'll define series:

Given $\{a_n\}$, define $\sum_{n=p}^{q} a_n = a_p + \dots + a_q$

let $\boxed{S_n = \sum_{k=1}^{n} a_n}$ the $n^{th}$ partial sum.

The partial sums $\{S_n\}$ form a sequence, written $\sum_{k=1}^{\infty} a_k$, called an infinite series.

This may or may not converge, but if it does, say to $s \in X$, then we write $\sum_{n=1}^{\infty} a_n = s$.

Note: $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_n$.

* Thus, a series is a sequence of partial sums.

Question: When does a series converge? (i.e., when does the sequence of partial sums converge?)

Ex: $a_n = \frac{1}{n}$. Does $\underbrace{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots}_{\text{"Harmonic series"}}$ converge?

No! The sequence (of partial sums) isn't Cauchy.

Recall: $S_{2n} - S_n \geq \frac{1}{2}$.

Theorem (Cauchy criterion for series)

$\sum a_n$ converges $\iff$ $\boxed{\forall \varepsilon > 0,\ \exists N \text{ s.t. } m, n \geq N \Rightarrow \left|\sum_{k=n}^{m} a_k\right| < \varepsilon.}$

Proof: Exercise. (Nothing more than translating def'ns)

Cor: $\sum a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$.

[Just let $n=m$.] Note: Converse of Cor. fails.

**Theorem** (non-neg. series). If $a_n \geq 0$, then

$$\sum a_n \text{ converges} \iff \text{partial sums are bounded.}$$

Proof ($\Leftarrow$) If $a_n \geq 0$, partial sums are monotonically increasing, and they're bounded $\Rightarrow$ partial sums converge.

($\Rightarrow$) Obvious. $\square$

**Theorem**: (Comparison test)

ⓐ If $|a_n| \leq c_n$ (for $n$ large enough, say $n \geq M$) and $\sum c_n$ converges, then $\sum a_n$ converges.

ⓑ If $a_n > d_n \geq 0$ (for $n$ large enough, say $n \geq M$) and $\sum d_n$ diverges, then $\sum a_n$ diverges.

Proof: (a) Fix $\varepsilon > 0$. Since $\sum c_n$ converges, $\exists N_1$ s.t.

$$m, n \geq N_1 \Rightarrow \left|\sum_{k=m}^{n} c_k\right| < \varepsilon.$$

Let $N = \max\{N_1, M\}$.

Now, for $k \geq N$, $\left|\sum a_k\right| \leq \sum |a_k| \leq \sum c_k < \varepsilon$. ✓

(b) Use (a) to show: If $\sum a_n$ converges, so does $\sum d_n$. (Exercise)

$\square$

- What to compare to?

Geometric series: If $|x| < 1$, then $\boxed{\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}}$

If $|x| \geq 1$, then $\sum_{n=1}^{\infty} x^n$ diverges.

Proof: If $x \neq 1$, let $S_n = 1 + x + \dots + x^n = \frac{1-x^{n+1}}{1-x}$

Then $\lim_{n\to\infty} S_n = \frac{1}{1-x} \cdot \lim_{n\to\infty} 1 - x^{n+1} = \frac{1}{1-x}$ if $|x|<1$.

If $|x| \geq 1$, then $s_n \not\to 0$ so $\sum x^n$ diverges. $\square$

Question: Does $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p}}$ converge or diverge? For which $p$?

Theorem (Cauchy): If $a_1 \geq a_2 \geq a_3 \geq \dots \geq 0$. $a_1 + 2a_2 + 4a_4 + 8a_8 + \dots$

$\boxed{\text{Then } \sum a_n \text{ converges} \iff \sum 2^k a_{2^k} \text{ converges}}$

Proof ($\Leftarrow$) Suppose that $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Let $t_k = a_1 + 2a_2 + \dots + 2^k a_{2^k}$ and $S_n = a_1 + a_2 + \dots + a_n$.

If $n < 2^k$: $S_n \leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \dots + (\overbrace{a_{2^k} + \dots + a_{2^{k+1}-1}}^{\text{"extra"}})$

$\leq a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \dots + 2^k a_{2^k}$

$= a_1 + 2a_2 + 4a_4 + \dots + 2^k a_{2^k} = t_k$

Since $t_k$ converges, so does $S_n$. ✓

($\Rightarrow$) $2s_n = 2a_1 + 2a_2 + 2(a_3+a_4) + 2(a_5+a_6+a_7+a_8) + \dots$

$t_k = a_1 + (a_2+a_2) + 4a_4 + 8a_8 + \dots$

If $n > 2^k$, then $t_k \le 2s_n$. Use comparison test: $t_k$ monot., bounded by $\lim_{n\to\infty} 2s_n$, so $t_k$ converges. □

Application: $\sum \frac{1}{n^p}$ converges if $p>1$ and diverges if $p \le 1$.

Proof: If $p \le 0$, then $\frac{1}{n^p} \not\to 0$, so series diverges.

If $p > 0$, use $\sum_{k=0}^{\infty} 2^k a_{2^k} = \sum_{k=0}^{\infty} 2^k \cdot \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} 2^{(1-p)k}$ geometric series!

This converges $\Leftrightarrow$ $(1-p)k < 0$ $\Leftrightarrow$ $p > 1$. □

Example

- $\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots$ converges (define the limit as "e")

Why? It's partial sums are bounded by $3 = 1 + 1 + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right)$

Convergence is rapid: $e - s_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots$

$$< \frac{1}{(n+1)!}\underbrace{\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots\right)}_{>1} < \frac{1}{n!\,n}$$

Cor: e is irrational!

Proof: Suppose $e = \frac{m}{n} \in \mathbb{Q}$. Consider $e - s_n = \frac{m}{n} - \left(1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!}\right)$

Clearly, $0 < \underbrace{n!(e - s_n)}_{\text{This is an integer!}} < n!\left(\frac{1}{n!\,n}\right) = \frac{1}{n}$ ↯ □