MTHSC 453

Lecture 19 — Convergence of Series

<u>Theorem</u> (Root test): Given $\sum a_n$, let $\boxed{\alpha = \limsup \sqrt[n]{|a_n|}}$

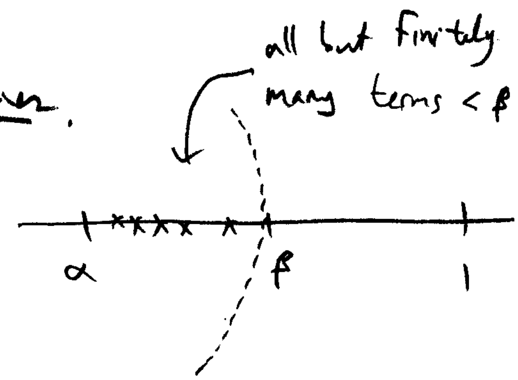
Unlike lim, the limsup always exists!

Then $\alpha < 1 \Rightarrow$ series <u>converges</u>

$\alpha > 1 \Rightarrow$ series <u>diverges</u>

$\alpha = 1 \Rightarrow$ test <u>inconclusive</u>.

<u>Proof</u>: By comparison with the <u>geometric series</u>.



- If $\alpha < 1$, choose β s.t. $\alpha < \beta < 1$.

  Then $\exists N$ s.t. $n \geq N \Rightarrow \sqrt[n]{|a_n|} < \beta$

  $$\Rightarrow |a_n| < \beta^n \text{ for } n \geq N$$

  But $\sum \beta^n$ converges $\Rightarrow \sum a_n$ converges ✓

- If $\alpha > 1$, then $\exists$ subsequence $\sqrt[n_k]{|a_{n_k}|} \longrightarrow \alpha > 1$.

  So $|a_{n_k}| > 1$ for infinitely many terms, so series diverges. ✓

- If $\alpha = 1$, note $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{n^2}$ converges, both have $\alpha = 1$. □

<u>Theorem</u> (Ratio test): Given $\sum a_n$, let $R = \limsup \left|\frac{a_{n+1}}{a_n}\right|$

Then $R < 1 \Rightarrow$ series converges

$R > 1 \Rightarrow$ series diverges

$R = 1 \Rightarrow$ test inconclusive.

In general, the root test is more powerful,

but the ratio test is usually easier.

Example: $1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \ldots$

Ratios sequence is $1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, \ldots$

Proof: (of ratio test). Comparison to geometric series

- $R<1$ $\quad \exists \beta$ s.t. $\left|\frac{a_{n+1}}{a_n}\right| < \beta < 1$ for all $n \geq N$.

So $|a_{n+1}| < \beta |a_n| < \beta^2 |a_{n-1}| < \beta^3 |a_{n-2}| < \ldots$

and $|a_{N+k}| < \beta^k a_N$.

Tail: $\sum_{k=0}^{\infty} a_{N+k} \leq a_N \underbrace{\sum \beta^k}_{\text{converges!}}$

- For $R > 1$, see terms $\not\to 0$ (exercise.) $\square$

Powerseries If $c_n \in \mathbb{C}$, $\sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + c_2 z^2 + \ldots$ is a power series.

Question: For what $z \in \mathbb{C}$ does it converge?

Theorem: If $\alpha = \limsup \sqrt[n]{|c_n|}$, let $R = \frac{1}{\alpha}$ ("radius of convergence")

$$\boxed{\begin{array}{l} \text{then } \sum c_n z^n \quad \text{converges if } |z| < R \\ \qquad\qquad\quad\ \ \text{diverges if } |z| > R \end{array}}$$

Note: This is a disk in $\mathbb{C}$!

Proof: Use root test: $\sqrt[n]{|a_n|} = |z| \sqrt[n]{|c_n|}$ (i.e., $a_n = c_n z^n$)

Apply lim sup: $\limsup \sqrt[n]{|a_n|} = |z| \limsup \sqrt[n]{|c_n|} := \frac{|z|}{R}$. □

Question: Given two series $\{a_n\}, \{b_n\}$, what can we say about $\sum a_n b_n$?

Ans. "Summation by parts":

Let $A_n = \sum_{k=0}^{n} a_k$ for $n=0$, and set $A_{-1} = 0$

Then $$\boxed{\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_p b_q - A_{p-1} b_p} \quad (*)$$

Reminiscent of "integration by parts": $\int dv\, u = \int v\, du - vu$.

$\int \longleftrightarrow \sum \qquad u \longleftrightarrow b_n \qquad v \longleftrightarrow A_n$

$du \longleftrightarrow b_n - b_{n+1} \qquad dv \longleftrightarrow a_n$

Algebraic proof is straightforward. (check both sides)

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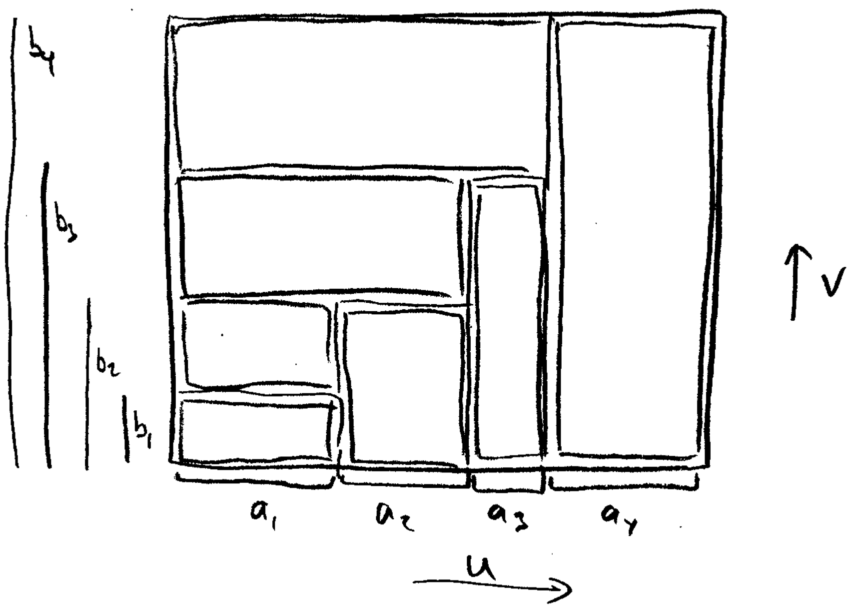
Geometric proof: (sketch)



$$\underbrace{\int u\,dv}_{\text{vertical boxes}} = \underbrace{uv}_{\text{total area}} - \underbrace{\int v\,du}_{\text{horizontal boxes}}.$$

Theorem: If A_n bounded, b_n positive and decreasing $\to 0$.

then $\sum a_n b_n$ converges.

Proof (idea). Say $|A_n| \le M$.

this works

Given $\varepsilon > 0$, $\exists N$ s.t. $b_N \le \frac{\varepsilon}{2M}$

For $q \ge p \ge N$,

$$\left|\sum_{n=p}^{q} a_n b_n\right| = \left|\sum A_n (b_n - b_{n+1}) + A_p b_q - A_{p-1} b_q\right|$$

$$\le M \left|\sum (b_n - b_{n-1}) + b_p + b_q\right|$$

$$\le 2 M b_p \le 2 M b_N \le \varepsilon. \quad \checkmark \qquad \square$$

Cor: If $|c_1| \ge |c_2| \ge \dots$, alternating signs and $\to 0$. Then $\sum c_n$ converges

Proof: $a_n = (-1)^{n+1}$, $b_n = |c_n|$.

<u>Sums</u> of series: $\sum a_n + \sum b_n = \sum a_n + b_n$

<u>Products</u> of series: $\left(\sum a_n\right)\left(\sum b_n\right) = \sum c_n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

Motivation: $(a_0 + a_1 z + a_2 z^2 + \dots)(b_0 + b_1 z + b_2 z^2 + \dots)$

$$= (a_0 b_0) + (a_1 b_0 + a_0 b_1) z + (a_2 b_0 + a_1 b_1 + a_0 b_2) + \dots$$

<u>Problem</u>: $\sum c_n$ may not converge, even if $\sum a_n$ & $\sum b_n$ do!

But it <u>does</u> converge if $\sum a_n$ & $\sum b_n$ converge <u>absolutely</u>.

Absolute convergence

<u>Def</u>: A series $\sum a_n$ <u>converges absolutely</u> if $\sum |a_n|$ converges.

<u>Ex</u>: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} +/- \dots$ converges, but not absolutely.

<u>Theorem</u>: If $\sum a_n$ converges absolutely, then it converges.

<u>Proof</u> (sketch) $\left|\sum_{k=n}^{m} a_k\right| \le \sum_{k=n}^{m} |a_k|$ ← small by Cauchy Criterion for $\sum |a_n|$.

Rearrangements

Remarkably, if a series $\sum a_n$ converges to $A \in \mathbb{R}$, it may be possible to <u>rearrange</u> its terms and wind up with a series that converges to something else, or even diverges!

Ex: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+/-\cdots = \ln 2.$

Theorem (Riemann)
- If $\sum a_n$ converges but not absolutely, then a rearrangement can have any limsup and liminf you like.
- If it converges absolutely, all rearrangements have the same limit.

Ex (cont.)

Odd terms (positive): $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots$

Even terms (negative): $-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}-\frac{1}{10}$

To converge to π:

$$\underbrace{\left(1+\frac{1}{3}+\frac{1}{5}+\cdots\right)}_{>\pi}\underbrace{-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\cdots}_{<\pi}+\underbrace{\cdots}_{>\pi}-\underbrace{\quad\quad}_{<\pi}+/-\cdots$$