MTHSC 453

Lectures 20 & 21 — Limits & Continuity

Functions: Let X, Y be metric spaces and $f: X \to Y$.

Recall: We know what this means: $\boxed{\lim_{n\to\infty} x_n = x}$

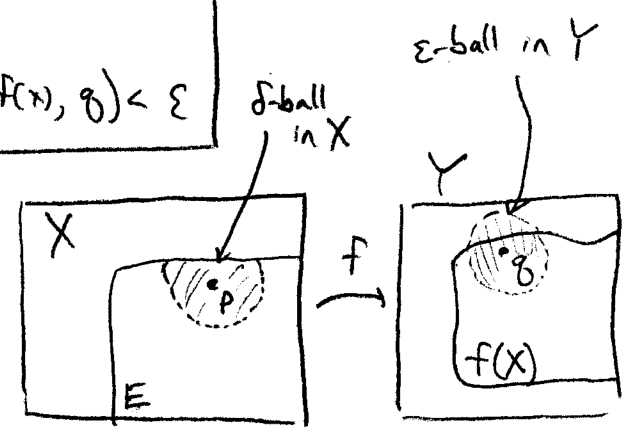
But what about $\boxed{\lim_{x\to p} f(x) = q}$? Does this make sense in any metric space?

Def: For $E \subset X$, $p \in X$ a limit pt. of E, let $f: E \to Y$.

To say $\boxed{\text{"}f(x) \to q \text{ as } x \to p\text{"}}$ or $\boxed{\text{"}\lim_{x\to p} f(x) = q\text{"}}$

means $\boxed{\begin{array}{l}\exists\, q \in Y \text{ s.t. } \forall \varepsilon > 0,\ \exists\, \delta > 0 \text{ s.t.} \\ \forall x \in E,\ 0 < d_X(x,p) < \delta \Rightarrow d_Y(f(x), q) < \varepsilon\end{array}}$

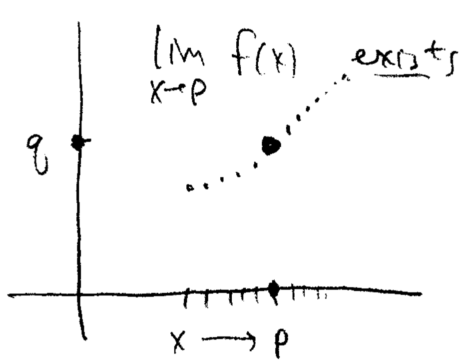
excludes $x = p$!

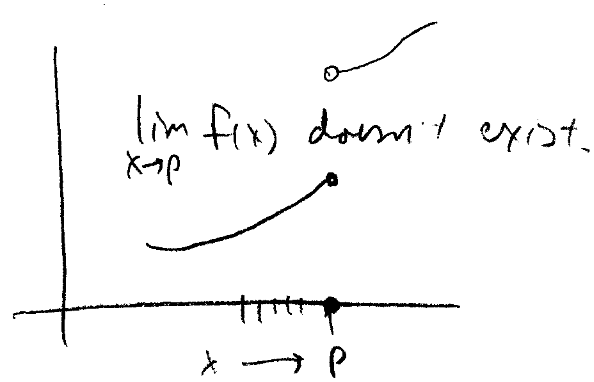


* Note that $0 < d_X(x,p) < \delta \Rightarrow d_Y(f(x),q) < \varepsilon$

is equivalent to $\boxed{f(N_\delta(p) \setminus \{p\}) \subset N_\varepsilon(q)}$

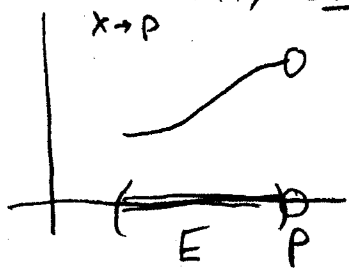
Examples





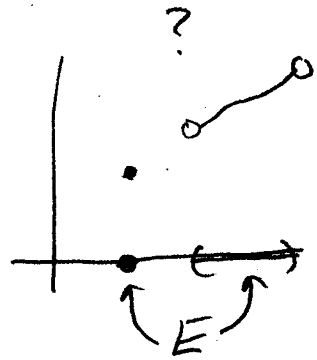
Ex:

$\lim_{x \to p} f(x)$ exists

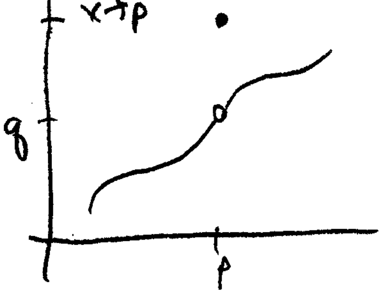


p doesn't have to be in E

?



$\lim_{x \to p} f(x)$ DNE

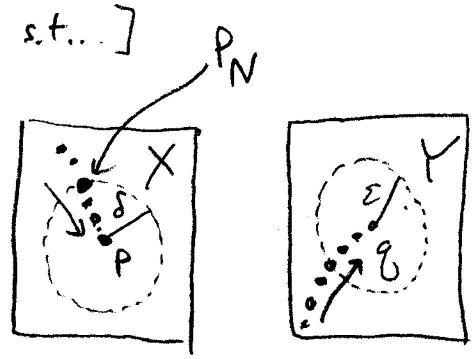


Theorem: $\boxed{\lim_{x \to p} f(x) = q} \iff \boxed{\forall \text{ seq. } \{p_n\} \text{ in } E,\ p_n \neq p \text{ and } p_n \to p,\ \text{we have } f(p_n) \to q}$

* This says the continuity preserves limits.

Proof: ($\Rightarrow$) Fix $\varepsilon > 0$. [Goal: Find N s.t. ...]

Since $\lim_{x \to p} f(x) = q$, $\exists\ \delta > 0$ s.t.



(*) $0 < d(x,p) < \delta \Rightarrow d(f(x), q) < \varepsilon$.

So, for a given $\{p_n\} \to p$, $\exists N$ s.t. $n \geq N \Rightarrow d(p_n, p) < \delta$.

So $n \geq N \Rightarrow d(f(p_n), q) < \varepsilon$, by (*). ✓

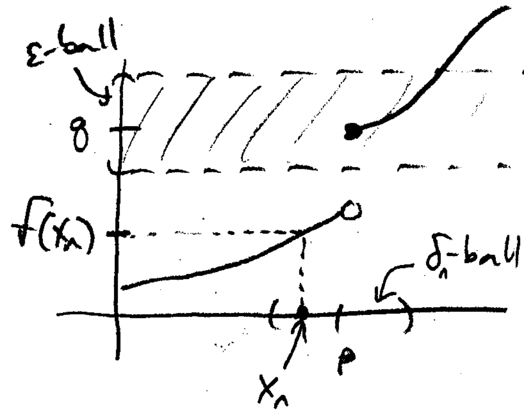
($\Leftarrow$) Suppose $\boxed{\lim_{x \to p} f(x) \not\to q}$. Then $\boxed{\exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0,\ \exists x \in E \text{ s.t. } 0 < d(x,p) < \delta \text{ but } d(f(x), q) \geq \varepsilon}$

Goal: We'll propose a "bad" sequence $\{p_n\}$ s.t. $f(p_n) \not\to q$.

Let $\delta_n = \frac{1}{n}$. Pick $x_n \in N_{\delta_n}(p)$ s.t. $d(f(x_n), q) \geq \varepsilon$.

Then $x_n \to p$ but $f(x_n) \not\to q$.

$\square$



<u>Benefit</u> of this: We know a lot about <u>sequences</u>. These results carry over!

<u>Recall</u>:
- Seq. limits are <u>unique</u>
- Limits of sums are sums of lim [$\lim(f+g) = \lim f + \lim g$]

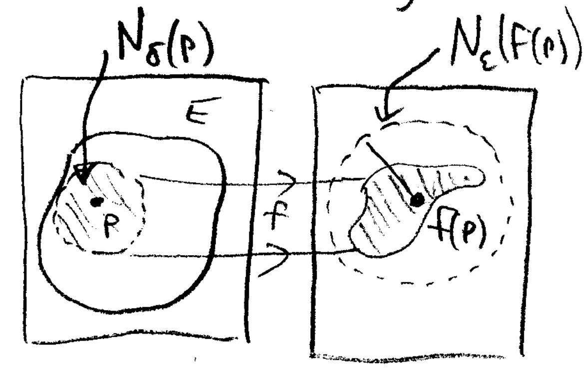
Continuous functions

<u>Def</u>: Let X, Y be metric spaces, $p \in E \subset X$, $f: E \to Y$.

Say f is <u>continuous at p</u> if $\boxed{\begin{array}{l} \forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. } \forall x \in E, \\ d(x,p) < \delta \Rightarrow d(f(x), f(p)) < \varepsilon \end{array}}$

<u>Remark</u>: The last line of this is equivalent to saying

$$\boxed{f(N_\delta(p)) \subset N_\varepsilon(f(p))}$$



i.e., $\exists$ δ-ball that gets maps <u>into</u> into our ε-ball

Theorem: If p is a limit point of E, then

$$\boxed{f \text{ is contin. at } p} \iff \boxed{\lim_{x \to p} f(x) = f(p)}$$

(This is immediate from the def'n of continuity.)

* If $\{x_n\}$ is a conv. seq., then $\boxed{f \text{ contin.} \iff \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right)}$

Sometimes we write $x_n \to x \implies \lim_{n\to\infty} f(x_n) = f(x)$,

i.e., continuous functions preserve limits.

Remark: Every function from $(\mathbb{R}, \text{disc}) \to Y$ is continuous! (why?)

Cor: Sums & products of continuous functions $X \to \mathbb{R}$ are contin.

Cor: $f, g : X \to \mathbb{R}^k$ s.t. $f = (f_1, \dots, f_k)$. Then

(a) f contin. $\iff$ each f_i is contin.

(b) $f+g$ & fg contin.

Proof: (Idea).

(a) Use $|f_i(x) - f_i(p)| \le \|f(x) - f(p)\| = \sqrt{\sum_{i=1}^{\infty} |f_i(x) - f_i(p)|^2}$

(b) Use components, & Part (a). $\square$

- <u>Summary</u> so far: We have 2 characterizations of continuity:

① In terms of <u>limits</u>:

For each $p \in X$, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $d(x,p) < \delta \Rightarrow d(f(x), f(p)) < \varepsilon$.

"close points map to close points"

② In terms of <u>sequences</u>:

For every conv. seq. $\{x_n\}$ in X, $\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right)$

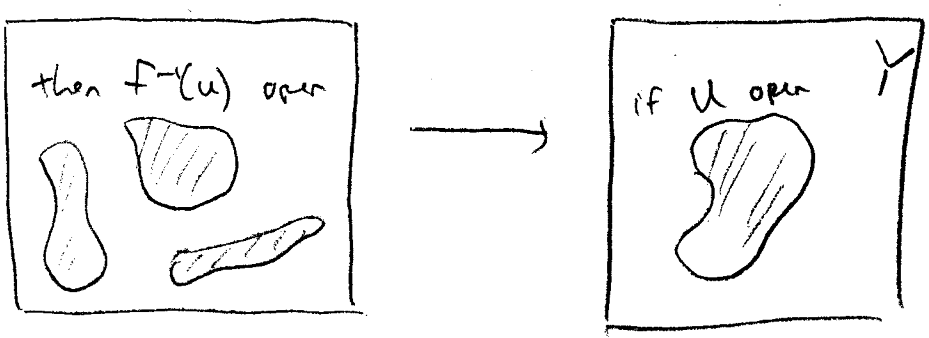
"Continuous functions preserve <u>limits</u> of <u>sequences</u>"

Remarkably, we have a third, <u>topological</u> characterization!

<u>Theorem</u>: $\boxed{f: X \to Y \text{ is contin.}} \iff \boxed{\begin{array}{l}\forall \text{ open sets } U \text{ in } Y, \\ f^{-1}(U) \text{ is open in } X\end{array}}$

Recall that $f^{-1}(U) = \{x \in X : f(x) \in U\}$, the "inverse image" of U

or "preimage" of U.



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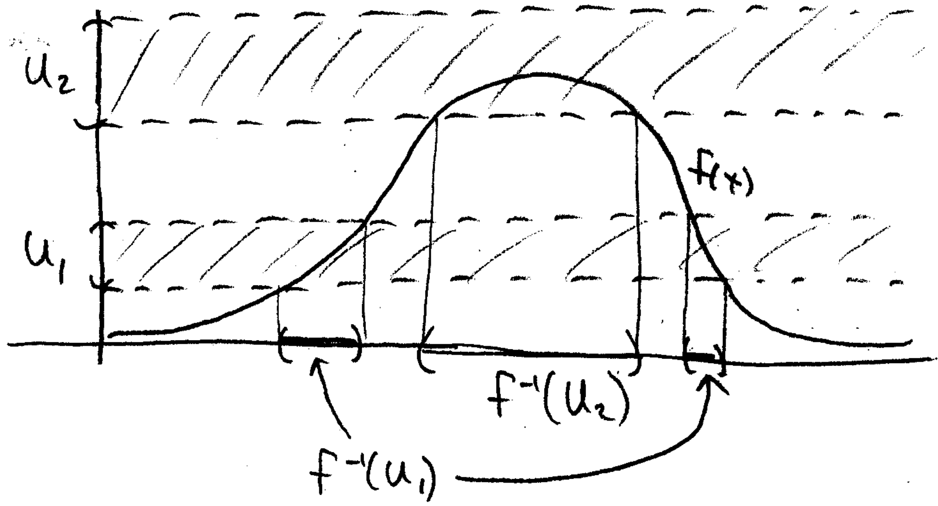
Examples

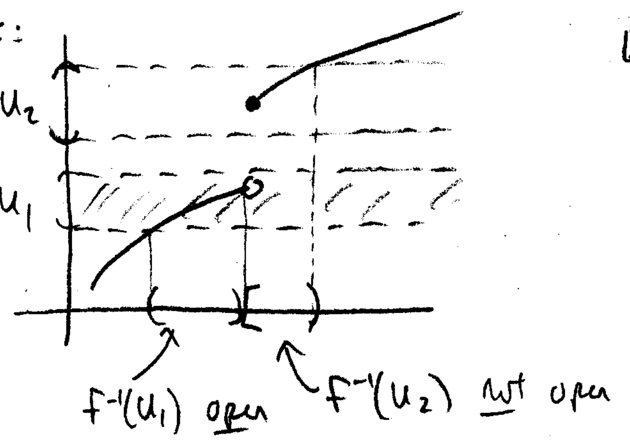


Note: $f: \mathbb{R} \to \mathbb{R}$ is continuous, but

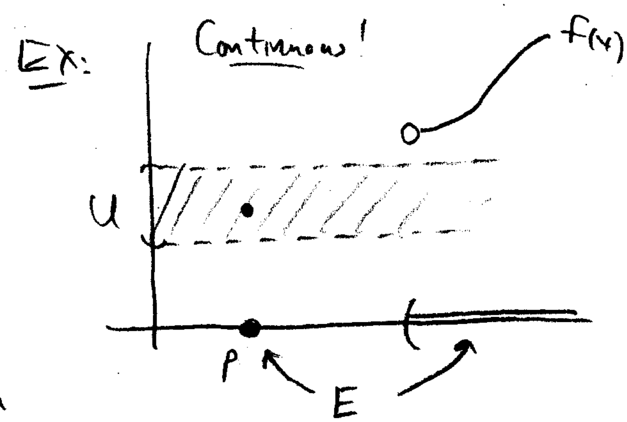
$$f(\mathbb{R}) = (0, M]$$

$\mathbb{R}$: open; $(0, M]$: not open!

Non-ex:



Ex: Continuous!



$f^{-1}(U) = \{p\}$, open $\boxed{\text{in } E}$!

because $N_\delta(p) = \{e \in E : d(e,p)\} = \{p\}$ for small δ.

Examples: Let $\mathbb{R}_{us} = (\mathbb{R}, \text{usual})$, $\mathbb{R}_{disc} = (\mathbb{R}, \text{discrete})$

every set is open!

$f: \mathbb{R}_{disc} \to \mathbb{R}_{us}$ $f(x) = x$ is continuous

$f: \mathbb{R}_{us} \to \mathbb{R}_{disc}$ $f(x) = x$ is not continuous.

$g: \mathbb{R}_{disc} \to \mathbb{R}_{us}$, $g(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$ is continuous!

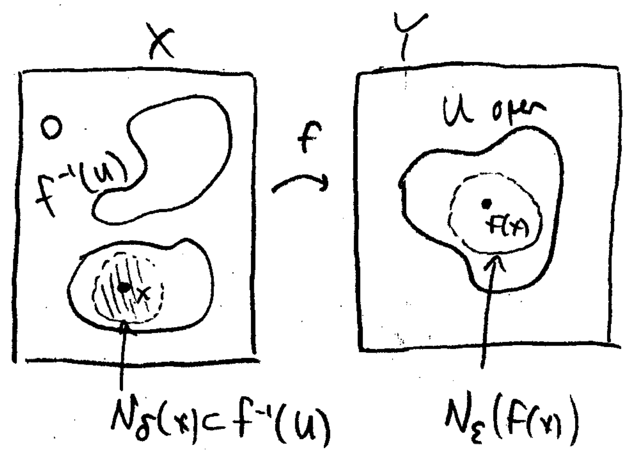
Why: $f^{-1}(U) = \begin{cases} \mathbb{Q} & U \text{ contains } 0, \text{ not } 1 \\ \mathbb{Q}^c & U \text{ contains } 1, \text{ not } 0 \\ \mathbb{R} & U \text{ contains } 0 \text{ and } 1. \end{cases}$

$g: \mathbb{R}_{us} \to \mathbb{R}_{disc.}$ (same g) is not continuous.

$$f^{-1}\Big(\big(-\tfrac{1}{2}, \tfrac{1}{2}\big)\Big) = \mathbb{Q}$$

$\uparrow$ open $\quad \uparrow$ not open in $\mathbb{R}_{us}$.

Proof (of theorem): $(\Rightarrow)$



Given U open in Y,

consider $x \in f^{-1}(U)$.

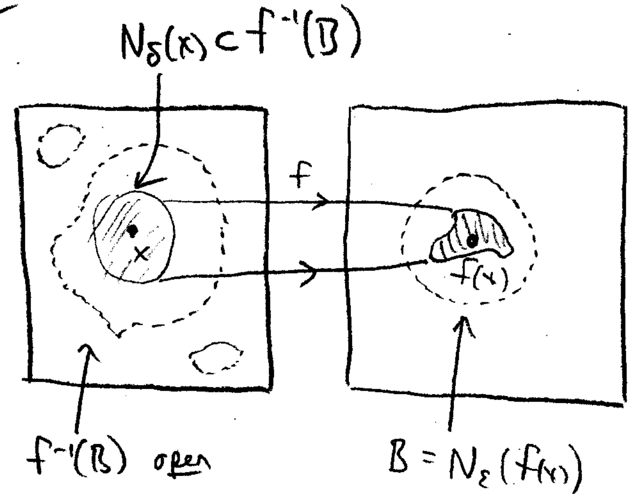
Goal: Show x is interior in $f^{-1}(U)$.

Note: $f(x) \in U$ which is open, so $\exists\, N_\varepsilon(f(x)) \subset U$.

By continuity of f, $\exists$ δ-ball $N_\delta(x)$ that maps into $N_\varepsilon(f(x)) \subset U$.

This means $N_\delta(x) \subset f^{-1}(U)$. ✓

$(\Leftarrow)$ Fix $x \in X$, $\varepsilon > 0$.



Let $B = N_\varepsilon(f(x))$.

Then $x \in f^{-1}(B)$, open by assumption.

Then x is an interior point of

$f^{-1}(B)$, so $\exists\, N_\delta(x) \subset f^{-1}(B)$.

But then $N_\delta(x)$ is mapped into $B = N_\varepsilon(f(x))$,

so f is continuous! ✓ $\square$

Consequences

Theorem:
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}, \quad f, g \text{ contin} \Rightarrow g\circ f \text{ contin.}$$

Proof: [Note that ε-δ proof is messy!]

Let $U \subset Z$ be open $\Rightarrow g^{-1}(U)$ is open in Y (b/c g contin.)

$\Rightarrow f^{-1}(g^{-1}(U))$ is open in X (b/c f contin.)

$\Rightarrow (g\circ f)^{-1}(U)$ is open in X. $\square$

Theorem: $\boxed{f: X \to Y \text{ contin}} \iff \boxed{\forall \text{ closed } K \text{ in } Y,\ f^{-1}(K) \text{ is closed in } X}$

Proof (idea)

$$f^{-1}(K) = [\underbrace{f^{-1}(\overset{\text{open}}{K^c})}_{\text{open}}]^c$$