MTHSc 453

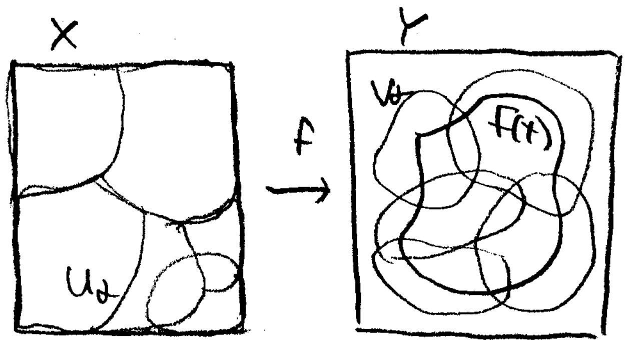
Lecture 22 Uniform continuity

* Continuous functions on compact sets enjoy many nice properties.

Theorem: If $f: X_{cpt} \longrightarrow Y$ is contin., then $f(X)$ is compact.

Proof: Let $\{V_\alpha\}$ be an open cover of $f(X)$, and let $U_\alpha := f^{-1}(V_\alpha)$.

Note: $\{U_\alpha\}$ is an open cover of X!



By compactness of X, $\exists$ finite subcover $\{U_1, \dots, U_n\}$.

Then $\{V_1, \dots, V_n\}$ is a finite subcover of $f(X)$. $\square$

Remark: It's possible to map $(a,b) \longrightarrow\!\!\!\!\!\rightarrow \mathbb{R}$, continuously

(e.g., $f(x) = \tan x$ maps $(-\pi/2, \pi/2) \longrightarrow\!\!\!\!\!\rightarrow \mathbb{R}$.)

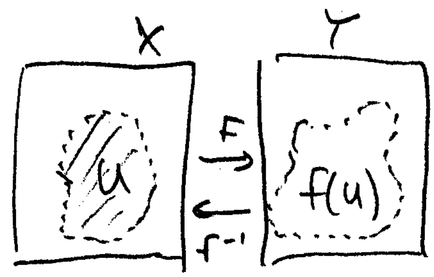
But it's not possible to map $[a,b] \longrightarrow\!\!\!\!\!\rightarrow \mathbb{R}$, continuously.

Cor If $f: X_{cpt} \rightarrow \mathbb{R}^k$ is contin., then $f(X)$ is closed & bounded.

Cor: If $f: X_{cpt} \rightarrow \mathbb{R}^k$ is contin., then f achieves its max. and min. values.

Theorem: Let $f: X \to Y$ be a contin. bijection. If X is cpt, then $f^{-1}: Y \to X$ is continuous.

[Such an f is called a homeomorphism.]



Proof: Let $U \subset X$ be open.

Goal: Show $f(U)$ is open.

Since U is open, U^c is closed

$\Rightarrow U^c$ is compact (b/c X is compact)

$\Rightarrow f(U^c)$ is compact (b/c f is contin, X cpt)

$\Rightarrow f(U^c)$ is closed (cpt sets are closed)

Thus, $f(U^c)^c = f(U)$ is open $\Rightarrow f^{-1}$ is contin. $\square$

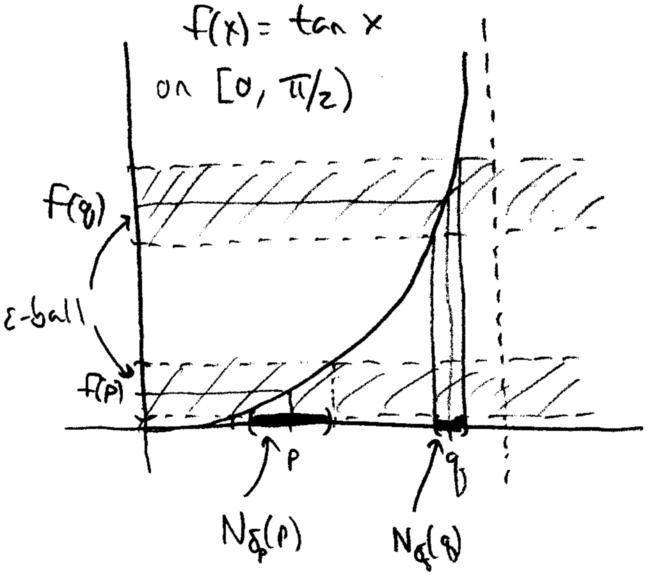
Def: Call $f: X \to Y$ uniformly continuous on X if

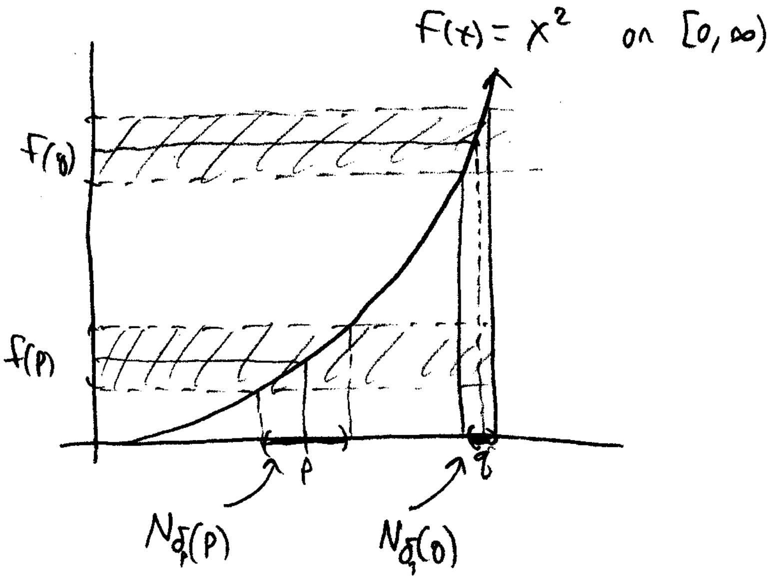
$$\boxed{\begin{array}{l} \forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. } \forall x, p \in X \\ d(x,p) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon \end{array}}$$

* Key difference: Given $\varepsilon > 0$, the same δ works for all $p \in X$.

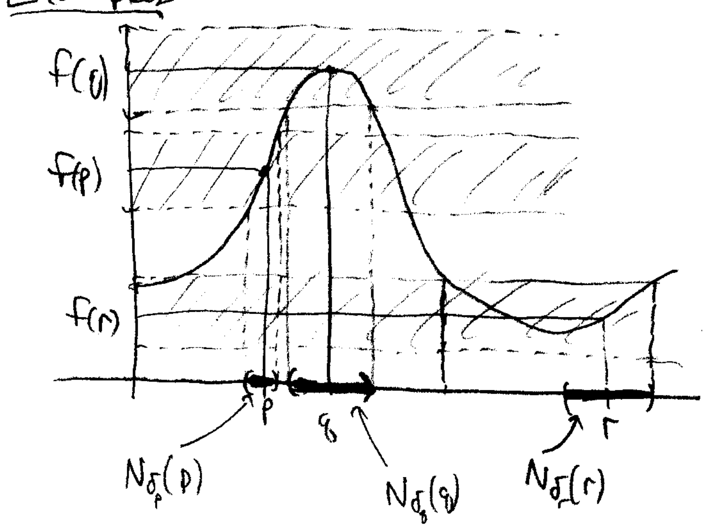
[If f is merely contin, then given $\varepsilon > 0$, each $p \in X$ has a (possibly different) δ_p that works.]

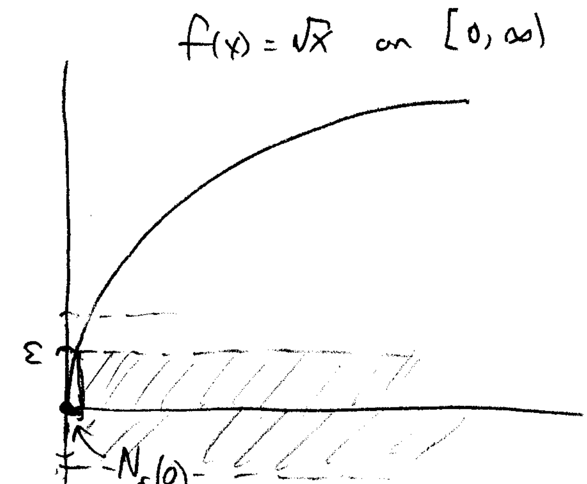
Non-examples





Examples





Theorem: If $f : X_{cpt} \to Y$ is contin, then f is uniformly contin.

Proof: Fix $\varepsilon > 0$. [Goal: Find δ that "works" for all $p \in X$.]

Each point $x \in X$ has a $\delta_x > 0$ s.t. $\boxed{d(x,p) < \delta_x \Rightarrow d(f(x), f(p)) < \frac{\varepsilon}{2}}$

<u>Why $\frac{\varepsilon}{2}$</u>: If $p, q \in N_{\delta_x}(x)$, then

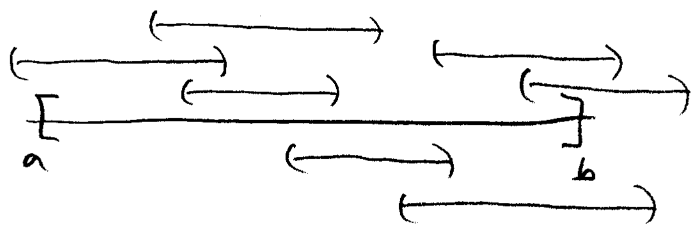
$$d(f(p), f(q)) \le d(f(p), f(x)) + d(f(x), f(q)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad (*)$$

Now, $\{N_{\delta_x}(x) : x \in X\}$ is an <u>open cover</u> of X.

Since X is compact, $\exists$ finite subcover $\{N_{\delta_{x_1}}(x_1), \ldots, N_{\delta_{x_n}}(x_n)\}$.

<u>Observe</u>: If there is some small $\delta > 0$ s.t. $d(p,q) < \delta \Rightarrow p \,\&\, q$ are in <u>some</u> $N_{\delta_{x_i}}(x_i)$, then by $(*)$, we're done!

<u>Claim</u>: Such a δ always exists.

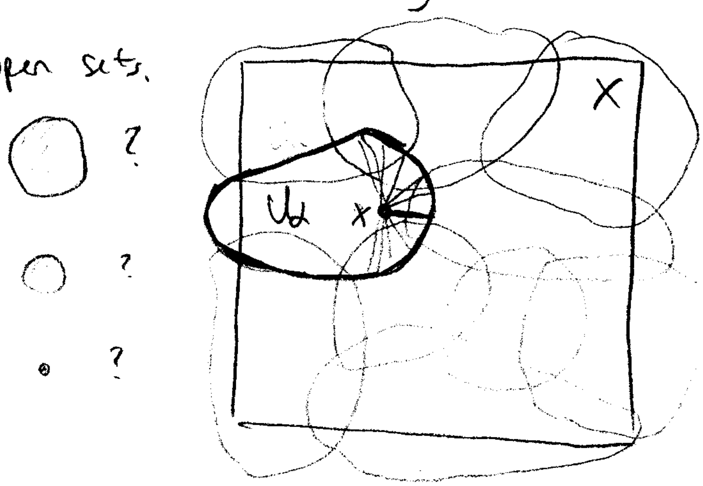


<u>Lebesgue covering lemma</u>: If $\{U_\alpha\}$ is an open cover of a <u>compact</u> set X, then $\exists\ \underbrace{\delta > 0}$ s.t. $\forall x \in X$, $B_\delta(x)$ is in some U_α.

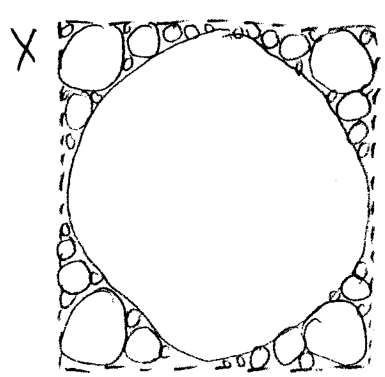
Called a "Lebesgue number" of the cover.

<u>Picture</u>: Put a circular sticker anywhere in X. <u>Claim</u>: It's entirely in one of the open sets.

Sticker:



<u>Non-example</u>

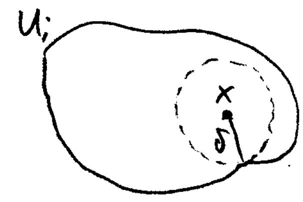


Proof: If $K \subset X$ is closed, define $d(x,K) = \inf\{d(x,y) : y \in K\}$

Claim: $d(x,K)$ is continuous in x. (Rudin, Exercise 4.20). ✓

Now, let $\{U_1, \dots, U_n\}$ be a finite subcover of $\{U_\alpha\}$.

Then $f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x, U_i^c)$ is continuous on X, so it attains its min. value, call it δ.



But $f(x) > 0$ (since if $x \in U_i$, $d(x, U_i^c) > 0$).

Thus for any $x \in X$, $f(x) \geq \delta \Rightarrow d(x, U_i^c) \geq \delta$ for at least one U_i^c. □

Remark: Proving this lemma establishes the theorem. □

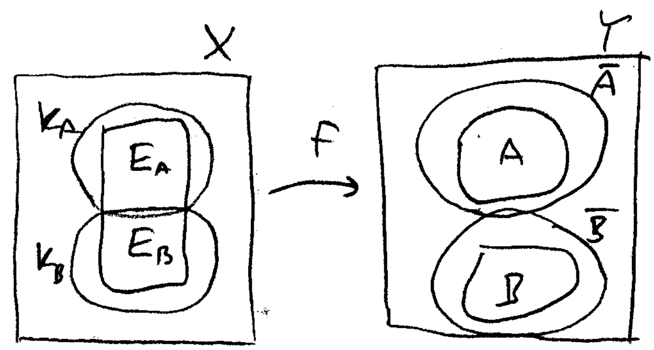
Theorem: If $f: X \to Y$ is continuous, $E \subset X$ is connected, then $f(E)$ is connected.

Proof: (Contrapositive)

Suppose $f(E)$ is disconnected.

Then $f(E) = A \cup B$ is a separation

$\bar{A} \cap B = A \cap \bar{B} = \emptyset$.



$E = E_A \cap E_B$ $\quad$ $f(E) = A \cup B$

Let $E_A = f^{-1}(A)$ $\quad$ $K_A = f^{-1}(\bar{A})$

$E_B = f^{-1}(B)$ $\quad$ $K_B = f^{-1}(\bar{B})$.

Note: $\begin{cases} E_A \cap K_B = \emptyset \\ E_B \cap K_A = \emptyset. \end{cases}$

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Since f is continuous, K_A and K_B are closed.

Thus, $E_A \subset \overline{E_A} \subset K_A$

$E_B \subset \overline{E_B} \subset K_B$

Clearly, $K_A \cap E_B = \emptyset \Rightarrow \overline{E_A} \cap E_B = \emptyset$

$K_B \cap E_A = \emptyset \Rightarrow E_A \cap \overline{E_B} = \emptyset$

Thus $E = E_A \cup E_B$ is a separation, hence E is disconnected. $\square$

Theorem: (Intermediate Value Theorem): If $f : [a,b] \to \mathbb{R}$ is contin., and $f(a) < c < f(b)$, then $\exists\, x \in (a,b)$ s.t. $f(x) = c$.

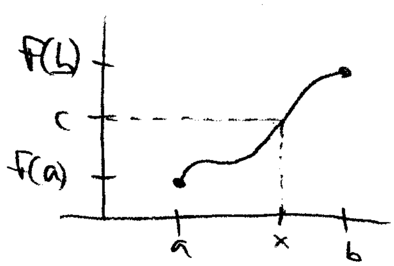
Proof. $[a,b]$ conn. $\Rightarrow f([a,b])$ conn, by Theorem.

But if $c \notin f([a,b])$, then

$\left((-\infty,c) \cap f([a,b])\right) \cup \left((c,\infty) \cap f([a,b])\right)$

would be a separation of $f([a,b])$,

which is impossible. $\square$



Remark. Converse is false!

$$f(x) = \begin{cases} 0 & x = 0 \\ \sin \frac{1}{x} & x \neq 0. \end{cases}$$

"topologist's sine curve"

This is not contin. at 0, but satisfies an IVP.