Lecture 23 Discontinuous Functions.

Dirichlet Function:

\[ F(x) = \begin{cases} 
1 & x \in \mathbb{Q} \\
0 & x \notin \mathbb{Q} 
\end{cases} \]

F is not continuous at any \( p \).

Example \( F(x) = \begin{cases} 
\sqrt{2} & x = \sqrt{2} \text{ (reduced)} \\
0 & x \notin \mathbb{Q} 
\end{cases} \)

Discontinuous at all rationals
Continuous at all irrationals. (Why?)

Discontinuities:

Let \( f: (a,b) \to \mathbb{R} \).

If for all \( \{x_n\} \text{ in } (x, 5) \) with \( x_n \to x \)
we have \( f(x_n) \to q \), write \( \lim_{x \to x^+} f(x) = q \) or \( \lim_{x \to x^+} f(x) = q' \).

Similarly, say \( f(x) = q' \) or \( \lim_{x \to x^-} f(x) = q' \).

If \( f \) is discontinuous but both of these limits exist but aren't equal, we say \( f \) has a simple discontinuity, or a discontinuity of the first kind.
Otherwise, we say $f$ has a discontinuity of the 2nd kind.

Example: $f(x) = \begin{cases} 0 & x = 0 \\ \sin x & x > 0 \end{cases}$

Continuous everywhere except at 0 (discont. of the 2nd kind).

Remark: In the \textit{1/3-Dirichlet} function, all discontinuities are simple.

Example: $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

Continuous at 0 (use e/5 argument)

Discontinuities of the 2nd kind at all other points. (Why?)

Monotonic Functions:

Recall that $f$ is monot. inc. if $x \leq y \Rightarrow f(x) \leq f(y)$.

and monot. decr. if $x < y \Rightarrow f(x) \geq f(y)$.

Theorem: If $f$ is monot. inc. on $(a,b)$, then $f(x^+)$ and $f(x^-)$ exist at every point.

[so $f$ has no discontinuities of the 2nd kind]
Proof: We have \( \sup_{t \in (x,x)} f(t) \leq f(x) \leq \inf_{t \in (x,x)} f(t) \). Call this \( A \).

Claim: \( A = f(x^+) \).

Given \( \varepsilon > 0 \), consider \( A - \varepsilon \).

\( \exists \delta \text{ s.t. } A - \varepsilon < f(x-\delta) \leq A \),

but then any \( t \in (x-\delta, x) \) but satisfies \( f(x-\delta) \leq f(t) \leq A \).

So \( f(t) \in (A - \varepsilon, A) \), as desired.

The \( f(x^+) \) case is analogous. \( \square \)

Theorem: If \( f \) is monotone on \((a,b)\), then the set of points where \( f \) is not continuous is countable.

Proof: \( \forall x \) when \( f \) is discontinuous, pick \( r(x) \in C \) s.t.

\( f(x^-) \leq r(x) \leq f(x^+) \)

set of discontinuities

By monotonicity, \( \forall x, y \in D \), then \( f(x) \neq f(y) \).

So we have an injection \( D \rightarrow \mathbb{Q} \). \( \square \)