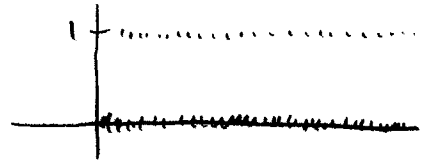
# MTHSc 453

## Lecture 23 Discontinuous Functions.

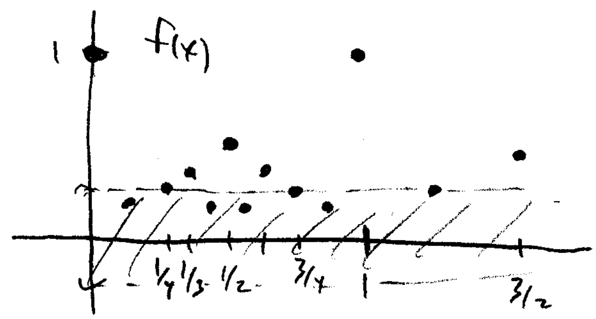
**Dirichlet function:**

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$




$f$ is not continuous at any $p$.

**Example** $f(x) = \begin{cases} 1/q & x = p/q \text{ (reduced)} \\ 0 & x \notin \mathbb{Q} \end{cases}$

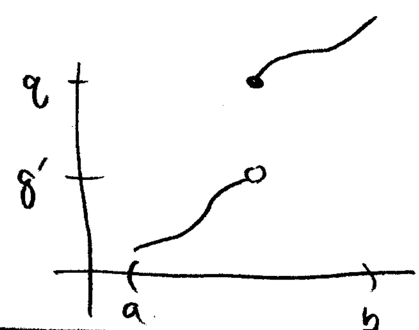



Discontinuous at all rationals

Continuous at all irrationals. (why?)

**Discontinuities:**

Let $f : (a,b) \to \mathbb{R}$.




If for all $\{t_n\}$ in $(x,b)$ with $t_n \to x$ we have $f(t_n) \to q$, write $\boxed{f(x^+) = q}$ or $\boxed{\lim_{t \to x^+} f(t) = q}$

Similarly, say $\boxed{f(x^-) = q'}$ or $\lim_{t \to x^-} f(t) = q'$.

If $f$ is discontinuous but both of these limits exist but aren't equal, we say $f$ has a <u>simple discontinuity</u>, or a discontinuity of the <u>first kind</u>.

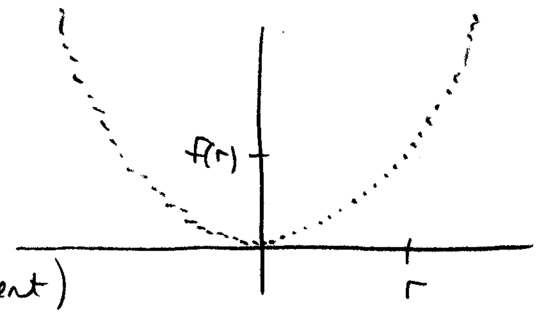
Otherwise, we say $f$ has a discontinuity of the <u>$2^{nd}$ kind</u>.

<u>Ex:</u> $f(x) = \begin{cases} 0 & x \le 0 \\ \sin\frac{1}{x} & x > 0 \end{cases}$

Continuous everywhere except at 0 (discont. of the $2^{nd}$ kind).

<u>Remark:</u> In the "$\frac{1}{q}$-Dirichlet" function, all discontins are <u>simple</u>.

<u>Ex:</u> $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$




Continuous at 0 (use $\varepsilon/\delta$ argument)

Discontinuities of the $2^{nd}$ kind at all other points. (Why?)

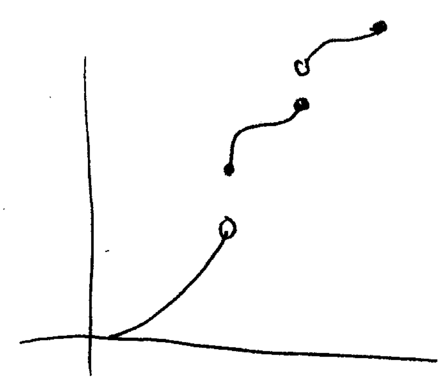
<u>Monotonic functions:</u>

Recall that $f$ is <u>monot. incr.</u> if $x \le y \Rightarrow f(x) \le f(y)$.

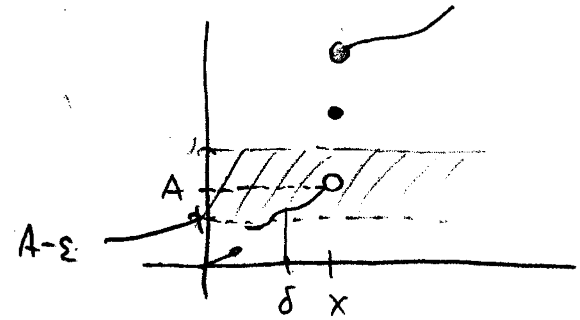
and <u>monot. decr.</u> if $x \le y \Rightarrow f(x) \ge f(y)$.

<u>Theorem:</u> If $f$ is monot. incr. on $(a,b)$, then $f(x^+)$ and $f(x^-)$ exist at every point.

[So $f$ has no discontinuities of the $2^{nd}$ kind]



Proof: We have $\underbrace{\sup_{t\in(a,x)} f(t)}_{\text{call this } A} \le f(x) \le \inf_{t\in(x,b)} f(t)$.




Claim: $A = f(x^-)$.

Given $\varepsilon > 0$, consider $A-\varepsilon$.

$\exists\ \delta$ s.t. $A-\varepsilon < f(x-\delta) \le A$,

but then any $t\in(x-\delta, x)$ but satisfies $f(x-\delta) \le f(t) \le A$.

So $f(t) \in (A-\varepsilon, A)$, as desired.

The $f(x^+)$ case is analogous. $\square$

Theorem: If $f$ is monotone on $(a,b)$, then the set of points where $f$ is not continuous is <u>countable</u>.

Proof: $\forall x$ where $f$ is discontinuous, pick $r(x) \in \mathbb{Q}$ s.t. $f(x^-) \le r(x) \le f(x^+)$.

set of discontinuities

By monotonicity, if $x, y \in D$, then $r(x) \ne r(y)$.

So we have an injection $D \hookrightarrow \mathbb{Q}$. $\square$