MTHSC 453

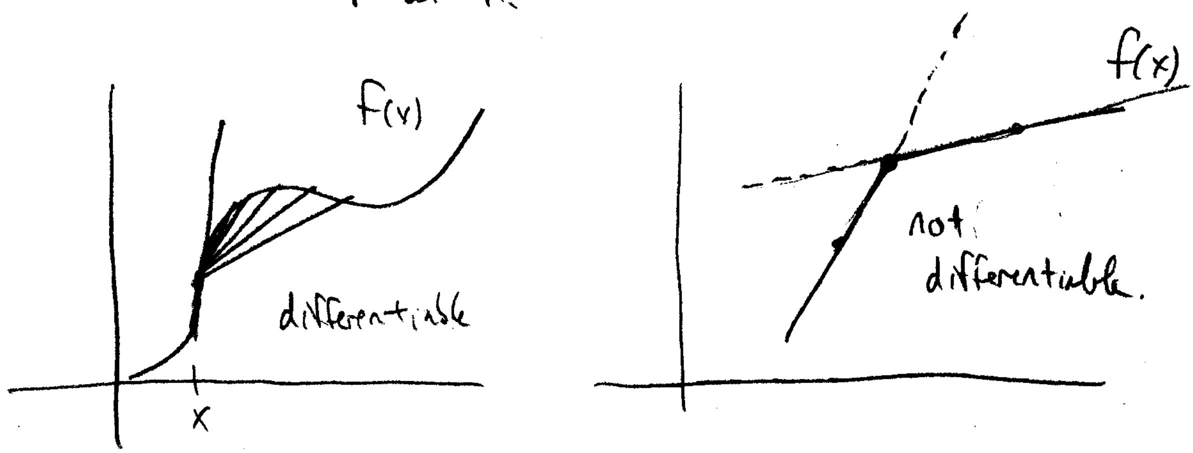
$\boxed{\text{Lecture 24 \& 25}}$ Differentiation

Def: A function $f:[a,b] \to \mathbb{R}$ is <u>differentiable</u> at $x \in [a,b]$ if the following limit exists:

$$\boxed{f'(x) := \lim_{t \to x} \underbrace{\frac{f(t)-f(x)}{t-x}}_{\text{slope of the secant line.}}} \qquad \begin{array}{l} t \in (a,b) \\ t \neq x \end{array}$$

The <u>derivative</u> of f at x.

<u>Example:</u>



<u>Question:</u> If f is <u>contin.</u> on $[a,b]$, is it diff'ble on $[a,b]$?

<u>Ans:</u> $\boxed{\text{No}}$ (See picture above)

<u>Question</u>: If f is <u>diff'ble</u> on $[a,b]$, is it <u>contin.</u> on $[a,b]$?

<u>Ans</u>: $\boxed{\text{Yes}}$. Proof: Show that $\lim_{t \to x} f(t) = f(x)$, i.e.,

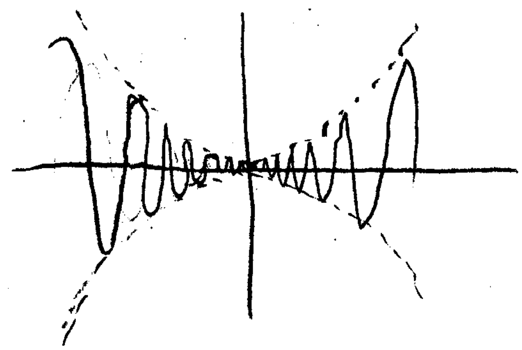
that $\lim_{t \to x} \left[f(t) - f(x) \right] = 0$

Note that $\lim_{t\to x} f(t)-f(x) = \underbrace{\frac{f(t)-f(x)}{t-x}}_{=f'(x)} \cdot \underbrace{(t-x)}_{=0} = 0.$ $\square$

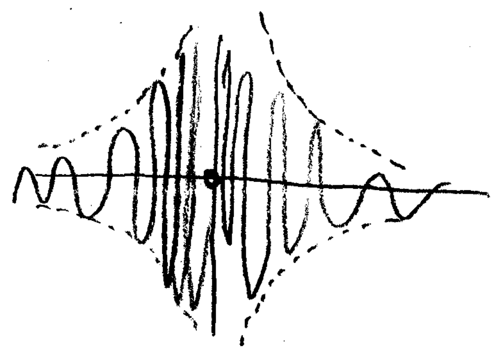
Question: If f is diff'ble on $[a,b]$, must f' be continuous?

Ans: $\boxed{\text{No.}}$

$$f(x) = \begin{cases} x^{4/3} \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$$



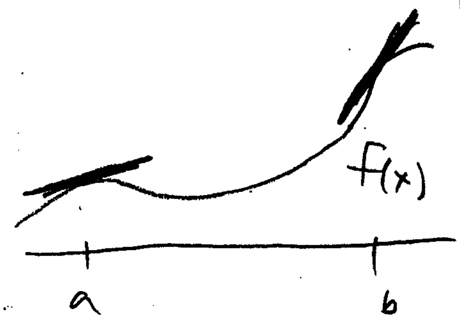
$$f'(x) = \begin{cases} \frac{4}{3} x^{1/3} \sin \frac{1}{x} - x^{-2/3} \cos \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



Remark: $f'(x)$ has a discontinuity of the second kind at $x=0$.

In fact, the following result shows that if f is diff'ble on $[a,b]$, then f cannot have any simple discontinuities on $[a,b]$.

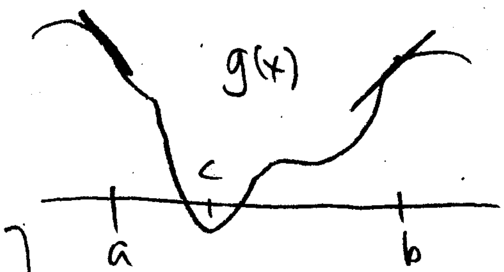
Theorem: Suppose $f:[a,b]\to\mathbb{R}$ is differentiable, and that $f'(a) < \lambda < f'(b)$. Then $\exists\, c \in (a,b)$ s.t. $f'(c) = \lambda$.



Proof: Put $g(t) = f(t) - \lambda t$.

Then $g'(a) < 0 \Rightarrow g(t_1) < g(a)$ for some $t_1 \in (a,b)$.

$g'(b) > 0 \Rightarrow g(t_2) < g(b)$ for some $t_2 \in (a,b)$.



Thus $g(x)$ has a local minimum at some $c \in (a,b)$.

At c, we have $g'(c) = 0$. [Proof later...]

$$\Rightarrow g'(c) = f(c) - \lambda = 0 \Rightarrow f(c) = \lambda. \quad \square$$

Def: A function f is a $\mathcal{C}^1$-function if f' exists and is contin., and is a $\mathcal{C}^k$-function if $f^{(k)}$ exists & is contin.

Note: $\mathcal{C}^0$: f is contin.

$\mathcal{C}^\infty$: All derivatives of f are contin. (Also called smooth.)

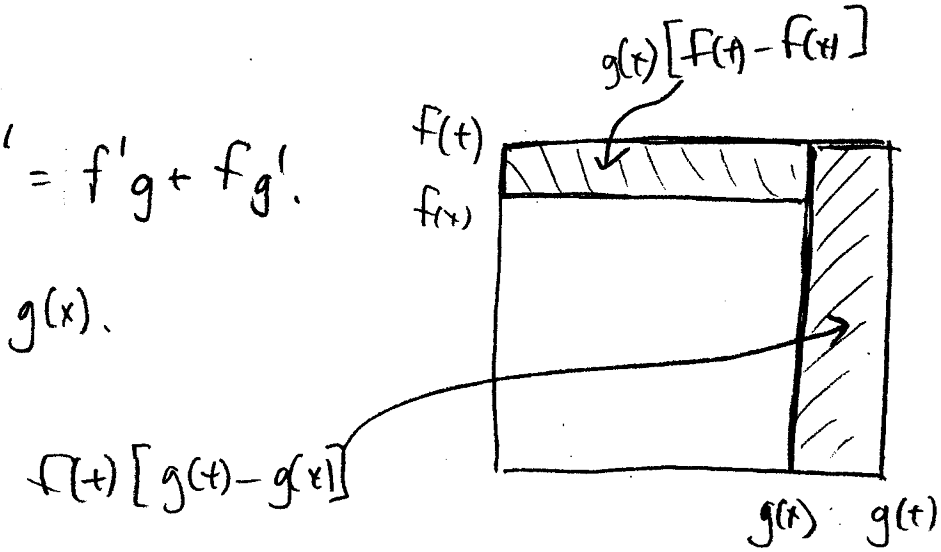
Remark: Since f' is a limit, then standard sum, product, and quotient rules can be derived.

Ex: $(f+g)' = f' + g'$ (b/c sum of a limit = limit of a sum).

<u>Product rule:</u> $(fg)' = f'g + fg'$.

<u>Proof:</u> let $h(x) = f(x)\,g(x)$.

See picture:



<u>Shaded area</u> $= h(t) - h(x) = f(t)[g(t) - g(x)] + g(t)[f(t) - f(x)]$

Divide by $t - x$ and take $\lim_{t \to x}$:

$$\Rightarrow \quad h'(x) = f(x)\,g'(x) + g(x)\,f'(x). \qquad \square$$

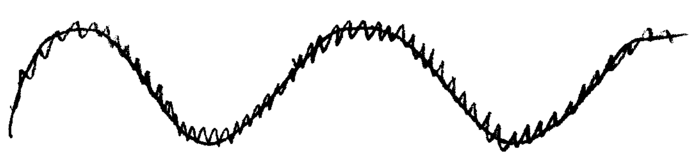
<u>Remark:</u> There exist functions $\mathbb{R} \to \mathbb{R}$ that are <u>continuous everywhere</u> but <u>diff'ble nowhere</u>.

<u>Example</u>: $f(x) = \sum_{n=1}^{\infty} a^n \cos(b^n x)$

$0 < a < 1$

$b = 2k + 1 \in \mathbb{Z}$

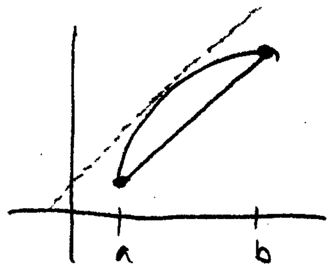
$ab > 1 + \frac{3\pi}{2}$



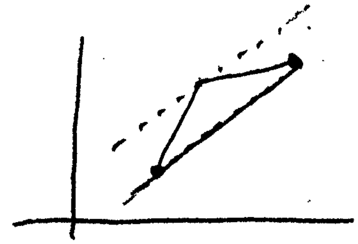
<u>Mean Value Theorem</u>

If f is contin. on $[a,b]$, and diff'ble on (a,b), then $\exists\, c \in (a,b)$ s.t. $f(b) - f(a) = (b-a)\,f'(c)$

The MVT connects the <u>value of f</u> to the <u>value of f'</u> without using limits.



vs.



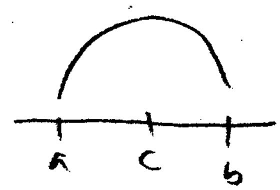
<u>Example application</u>: If $f'(x) > 0$ for all $x \in (a,b)$, then $f(b) > f(a)$.

<u>Proof</u>. $f(b) - f(a) \overset{\text{MVT}}{=} \underbrace{(b-a)}_{>0} \cdot \underbrace{f'(c)}_{>0} > 0$, as desired. $\square$

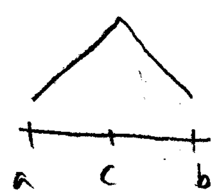
The following is a special case of the MVT:

<u>Theorem</u> (Simple version of "Rolle's theorem"): If $h : [a,b] \to \mathbb{R}$ is diff'ble & has a local <u>max</u> (or <u>min</u>) at $c \in (a,b)$, then $\boxed{h'(c) = 0}$.

<u>Proof</u>: Consider the quantity $\boxed{\dfrac{h(t) - h(c)}{t - c}}$ $(\ast)$



vs.



For $t_1 < t_2 < \dots \to c$, denomin. < 0

numer. < 0

so $(\ast) \geq 0$.

For $s_1 > s_2 > \dots \to c$, denom. < 0

numer. > 0, so $(\ast) \leq 0$.

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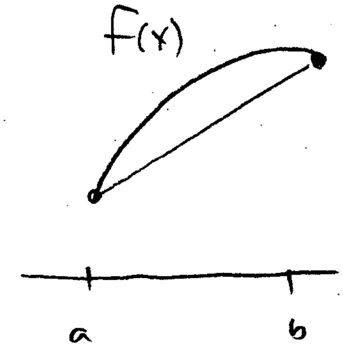
Thus, $\displaystyle\lim_{t_n \to c} \frac{h(t_n)-h(c)}{t_n - c} \geq 0$ & $\displaystyle\lim_{s_n \to c} \frac{h(s_n)-h(c)}{s_n - c} \leq 0$

This limit is $h'(c)$, and so $h'(c) = 0$. $\square$

(The case of local min is handled similarly.)



Remark: The MVT is a corollary.

Proof (Exercise).

Put $g(x) = f(x) - f(a) - \left[\dfrac{f(b)-f(a)}{b-a}\right](x-a)$,

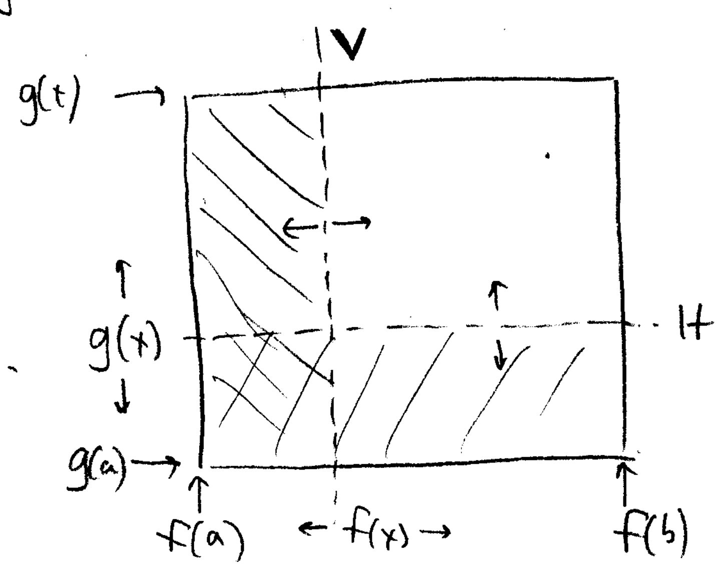
apply previous theorem. $\square$

Generalized MVT If $f(x)$, $g(x)$ contin on $[a,b]$
diff'ble on (a,b),
then $\exists\, c \in (a,b)$ s.t. $[f(b)-f(a)]\, g'(c) = [g(b)-g(a)]\, f'(c)$.

Remark: If $g(x) = x$, this is just the MVT.

Proof: (Motivating picture).
First, consider a "cake", and two knives V & H that are moving according to $\underbrace{f(x) \text{ \& } g(x)}$.

position



<u>Note:</u> LHS $= [f(b)-f(a)]\,g'(c)$ is the <u>rate</u> that H "sweeps" out <u>area</u>.

RHS $= [g(b)-g(a)]\,f'(c)$ is the rate that V "sweeps" out area.

This theorem says that at some point, the knives are sweeping out area at the same rate.

fudge factor... define this <u>later</u>.

Put $h(x) = [f(b)-f(a)]\,g(x) - [g(b)-g(a)]\,f(x) - \big(f(b)g(a)-f(a)g(b)\big)$

Note that $h(a) = [f(b)-f(a)]\,g(a) - [g(b)-g(a)]\,f(a) -$

$= \underline{f(b)g(a)} - \cancel{f(a)g(a)} - \underline{f(a)g(b)} + \cancel{f(a)g(a)} - \quad = 0$

And similarly, $h(b) = f(b)g(a) - \cancel{f(b)g(b)} - f(a)g(b) + \cancel{f(b)g(b)} - \quad = 0$

Since $h(a) = h(b) = 0$, [<u>Note:</u> was defined to force $h(a) = h(b) = 0$]

By previous theorem, $\exists\, c$ s.t. $h'(c) = 0$.

But $h'(c) = \text{LHS} - \text{RHS} = 0$ ✓ $\square$

<u>Taylor's theorem</u>

Suppose we know $f(a)$, and want to approximate $f(b)$.

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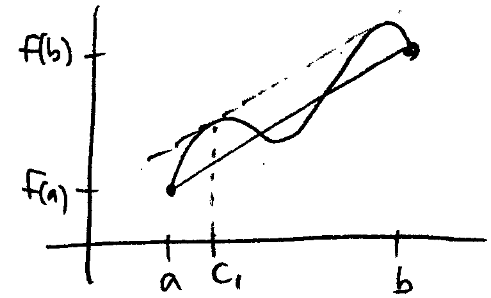
MVT says $f(b) = f(a) + \overbrace{f'(c_1)(b-a)}^{\text{"error term"}}$

for <u>some</u> $c_1 \in (a,b)$



We may not know c_1.

But suppose we know $f'(a)$.

How to find

Then we can write $f(b) = f(a) + f'(a)(b-a) + \text{error}$

In fact, Taylor says this is $\frac{f''(c_2)}{2}(b-a)^2$; a "better" error.

More generally, define $\boxed{P_{n-1}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}}$

<u>Taylor's Theorem</u> If $f^{(n-1)}$ contin. on $[a,b]$

$f^{(n)}$ exists on (a,b)

Then $P_{n-1}(x)$ <u>approximates</u> $f(x)$, and $f(b) = P_{n-1}(b) + \frac{f^{(n)}(c_n)}{n!}(b-a)^n$,

for some $c \in (a,b)$.

<u>Remark</u>: If $n=1$, this is $f(x) = f(a) + f'(c_1)(x-a)$, the <u>MVT</u>.

<u>Note</u>: $P_n(x)$ is the "best" polynomial approx. of order n at a,

because $f(a) = P_{n-1}(a)$, $f'(a) = P'_{n-1}(a)$, $\dots$, $f^{(n-1)}(a) = P^{(n-1)}_{n-1}(a)$.

Clearly, for some M, $\boxed{f(b) = P_{n-1}(b) + M(b-a)^n}$ $(*)$

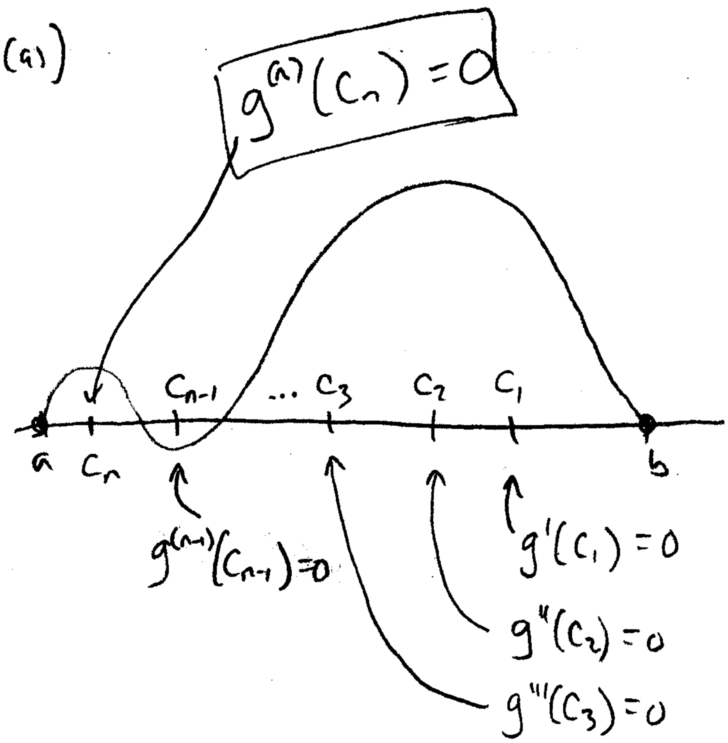
Let $g(x) = f(x) - P_{n-1}(x) - M(x-a)^n$

$\boxed{g^{(n)}(x) = f^{(n)}(x) - M \cdot n!}$ (why?)

Goal: Show $g^{(n)}(c) = 0$ for some $c \in (a, b)$. (why?)

Check:
$$\begin{cases} g(a) = 0 & (\text{since } f(a) = P_{n-1}(a)) \\ g'(a) = 0 & (\text{since } f'(a) = P'_{n-1}(a)) \\ g''(a) = 0 & \quad '' \\ \vdots & \quad \vdots \\ g^{(n-1)}(a) = 0 & \quad '' \end{cases}$$

And $g(b) = 0$ by $(*)$



So, $g(a) = g(b) = 0 \Rightarrow \exists\, c_1$

s.t. $g'(c_1) = 0$ (by MVT)

- $g'(a) = g'(c_1) = 0 \Rightarrow \exists\, c_2$ s.t. $g''(c_2) = 0$
- $g''(a) = g''(c_2) = 0 \Rightarrow \exists\, c_3$ s.t. $g'''(c_3) = 0$

$\vdots$

- $g^{(n-1)}(a) = g^{(n-1)}(c_{n-1}) = 0 \Rightarrow \exists\, c_n$ s.t. $g^{(n)}(c_n) = 0$.

This shows $\boxed{M = \dfrac{f^{(n)}(c_n)}{n!}}$, so $(*)$ gives Taylor's result. $\square$