# MthSc 453: Real Analysis (Summer I 2012)
# Midterm 1
# May 31, 2012

NAME: Key

## Instructions

- Exam time is 75 minutes.
- You may *not* use notes or books.
- Calculators are *not* allowed.
- **Show your work.** Partial credit will be given.

| Question | Points Earned | Maximum Points |
|---|---|---|
| 1 | | 20 |
| 2 | | 30 |
| 3 | | 20 |
| 4 | | 30 |
| **Total** | | **100** |

Student to your left:

Student to your right:

1. Let $F$ be a field endowed with an order $<$.

   (a) (6 points) Define what it means for $F$ together with $<$ to be an *ordered field*. You do *not* need to define what a field is or what an order is.

   The field operations preserve the order, i.e.,

   (i) $x < y \Rightarrow x+z < y+z \quad \forall z$

   (ii) $x < y \Rightarrow cx < cy \quad \forall c > 0$

   (b) (6 points) Define what it means for $F$ to have the *least upper bound property*.

   Every nonempty subset $A \subset F$ bounded above has a least upper bound $\alpha \in F$. (Need not be in $A$!)

   (c) (8 points) What can you say about the fields $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, regarding these properties? Why is $\mathbb{R}$ is some sense "special"?

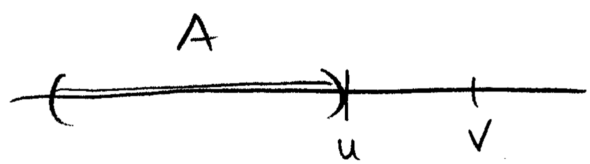
   Clearly, $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

   - $\mathbb{Q}$ does not have the l.u.b. property. (It has "holes")
   - $\mathbb{C}$ cannot be made into an ordered field (It is "too big.")
   - $\mathbb{R}$ is the only ordered field with the l.u.b. property!

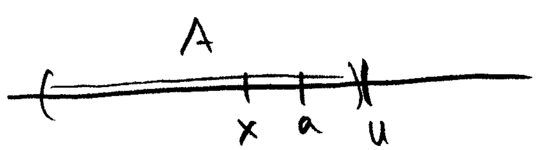
2. Let $u < \infty$ be an upper bound of a set $A \subset \mathbb{R}$.

   (a) (4 points each) Thus far, we've seen (at least) three equivalent definitions of the *supremum* of a set, which are started below. Complete the following three sentences, each of which is a condition from which we can conclude that $u = \sup A$.

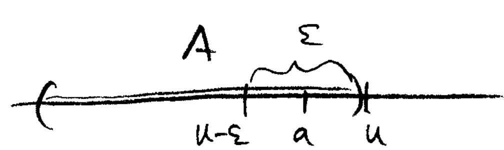
   (i) If $v \neq u$ is an upper bound of $A$, then ... $v > u$.



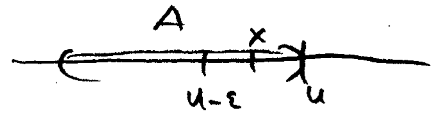

   (ii) If $x < u$, then ... $\exists\, a \in A$ s.t. $x < a \leq u$




   (iii) For any $\epsilon > 0$, ... $\exists\, a \in A$ s.t. $u - \varepsilon < a$



(b) (10 points) Pick *two* of your statements from Part (a) and prove that one implies the other. (You have six choices: (i)$\Longrightarrow$(ii), (ii)$\Longrightarrow$(iii), etc.)




(ii) $\Rightarrow$ (iii)

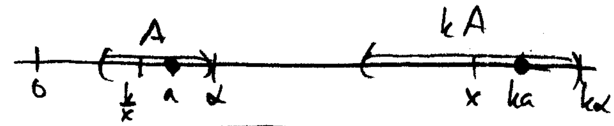
Suppose $u$ is an upper bound for $A$ s.t.

$\forall x < u,\ \exists a \in A$ s.t. $x < a \leq u$. $\leftarrow$ (ii)

Take $\varepsilon > 0$. Put $\boxed{x = u - \varepsilon}$. By (ii), $\exists\, a \in A$ s.t. $\underbrace{u - \varepsilon < a \leq u}_{\text{this is (iii)}}$. $\square$

(c) (6 points) For $k \in \mathbb{R}$, define the set $kA := \{ka : a \in A\}$. Prove that if $k \geq 0$, then $k \sup(A) = \sup(kA)$.

Let $\alpha = \sup A$. Assume $k > 0$ (the case of $k = 0$ is trivial).




Claim 1: $k\alpha$ is an upper bound for $kA$.

Proof: Pick $x \in kA$, say $x = ka$. $\boxed{\text{Since } a \leq \alpha \text{ \& } k \geq 0 \Rightarrow ka \leq k\alpha}$. ✓

Claim 2: $k\alpha$ is a least upper bound for $kA$.

Proof: Use def'n (a)(ii). Pick $x < k\alpha \Rightarrow \frac{x}{k} < \alpha$.

Since $\alpha = \sup A$, $\exists\, a \in A$ s.t. $\frac{x}{k} < a \leq \alpha \Rightarrow x < ka \leq k\alpha$. ✓ Hence $k\alpha = \sup kA$. $\square$

(d) (2 points) Does your proof hold for $k < 0$? Why or why not?

No. The "boxed" part requires $k \geq 0$.

3. (5 points each) For each of the following sets, decide if it is countable or uncountable. Give a *one sentence* justification for each.

   (a) The set $B_f = \{x_1, x_2, \ldots, x_n : x_i \in \{0,1\}, n \in \mathbb{N}\}$. That is, the set of *binary* sequences of *finite* length.

   Countable. It's a countable union of finite sets (and clearly infinite.)

   (b) The set $B_i = \{x_1, x_2, x_3, \ldots : x_i \in \{0,1\}\}$. That is, the set of *binary* sequences of *infinite* length.

   Uncountable. There is a bijection $B_i \longrightarrow 2^{\mathbb{N}} \not\sim \mathbb{N}$.

   (c) The set $Z_f = \{x_1, x_2, \ldots, x_n : x_i \in \mathbb{Z}, n \in \mathbb{N}\}$. That is, the set of *integer* sequences of *finite* length.

   Countable. It's a countable union of countable sets.

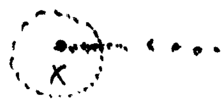
   (d) The set $Z_i = \{x_1, x_2, x_3, \ldots : x_i \in \mathbb{Z}\}$. That is, the set of *integer* sequences of *infinite* length.

   Uncountable. It contains $B_i$ (an uncountable set) as a subset.

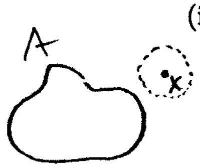
4. Let $(X, d)$ be a metric space, and $A \subset X$.

   (a) (4 points each) Carefully complete the following definitions:

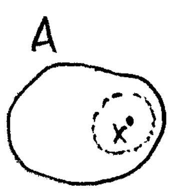
      (i) A point $x \in X$ is a *limit point* of $A$ if ... $\forall r \in \mathbb{R}$, $N_r(x)$ contains some $a \neq x$ in $A$.




      (ii) A point $x \in A$ is an *isolated point* of $A$ if ... it is <u>not</u> a limit point of $A$.




      (iii) A point $x \in A$ is an *interior point* of $A$ if ... $\exists r \in \mathbb{R}$ s.t. $N_r(x) \subset A$.




   (b) (9 points) Now, suppose that $X = \mathbb{R}$, and $d(x, y) = |x - y|$ (that is, the *Euclidean* metric). Write down the limit points, isolated points, and interior points of each of the following sets: $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. (No proofs needed.)

| | $\mathbb{Z}$ | $\mathbb{Q}$ | $\mathbb{R}$ |
|---|---|---|---|
| limit pts | none | $\mathbb{R}$ | $\mathbb{R}$ |
| isolated pts | $\mathbb{Z}$ | none | none |
| interior pts | none | none | $\mathbb{R}$ |

(c) (9 points) Finally, consider the *discrete metric* on $\mathbb{R}$, defined as

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

Again, characterize the limit points, isolated points, and interior points of each of the following sets: $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. (No proof needed.)

| | $\mathbb{Z}$ | $\mathbb{Q}$ | $\mathbb{R}$ |
|---|---|---|---|
| limit pts | none | none | none |
| isolated pts | $\mathbb{Z}$ | $\mathbb{Q}$ | $\mathbb{R}$ |
| interior pts | $\mathbb{Z}$ | $\mathbb{Q}$ | $\mathbb{R}$ |

* Note that for any $x \in \mathbb{R}$, $N_{1/2}(x) = \{x\}$, so every point is both isolated and interior of every nonempty set.