MthSc 453: Real Analysis (Summer I 2012)
Midterm 2
June 15, 2012

NAME: Key

Instructions

- Exam time is 75 minutes.
- You may *not* use notes or books.
- Calculators are *not* allowed.
- **Show your work.** Partial credit will be given.

Question	Points Earned	Maximum Points
1		14
2		16
3		20
4		25
5		10
6		15
Total		**100**

Student to your left:

Student to your right:

1. (14 points) Let X be a metric space. Recall that a set G is *open* in X if every point of G is an interior point, and that a set F is *closed* in X if it contains all of its limit points. The following classic theorem describes how open and closed sets are related:

A set E is open if and only if its complement E^c is closed.

Prove *one direction* of this theorem. (That is, prove (i) E open $\Longrightarrow$ E^c closed, or prove (ii) E closed $\Longrightarrow$ E^c open.)

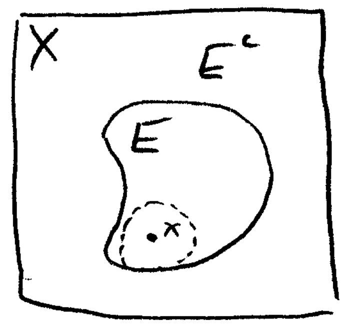
($\Rightarrow$) Suppose E is open, and take $x \in E$.

Then $\exists\, N_r(x) \subset E$, since x is interior.

This means $N_r(x) \cap E^c = \emptyset$, thus

x is not a limit point of E^c.

So E^c contains all of its limit points $\Rightarrow$ E^c is closed. $\square$



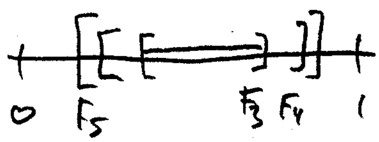
2. (16 points) Let X be a metric space. For each of the following statements, decide if it is true of false. For each false statement, give an explicit example showing how can fail.

(a) For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.

True.

(b) For any collection $\{F_\alpha\}$ of closed sets, $\bigcup_\alpha F_\alpha$ is closed.

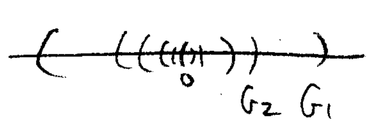
False. Consider $F_n = [\frac{1}{n}, 1 - \frac{1}{n}]$ for $n = 3, 4, 5, \ldots$



Then $\bigcup_{n=3}^{\infty} F_n = (0,1)$

(c) For any collection $\{G_\alpha\}$ of open sets, $\bigcap_\alpha G_\alpha$ is open.

False. Consider $G_n = (-\frac{1}{n}, \frac{1}{n})$ for $n = 1, 2, 3 \ldots$



Then $\bigcap_{n=1}^{\infty} G_n = \{0\}$.

(d) For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.

True.

3. (20 points) Let $X = \mathbb{R}$.

(a) Prove that the set $K = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ is compact, without using the Heine-Borel theorem. (That is, prove that every open cover of K has a finite subcover.)

Let $\{U_\alpha\}$ be an open cover of K, and say U_0 contains 0. Then $\exists\, N_r(0) \subset U_0$ for some r, and so if $n > \frac{1}{r}$, $\frac{1}{n} \in N_r(0)$.

Thus U_0 contains all but finitely many points of K, so there is clearly a finite subcover.

[For each $x_i \in K \cap U_0^c$, take some U_i containing x_i. Then $\{U_0, U_1, \dots, U_n\}$ is a finite subcover.] $\square$

(b) Prove that the set $A = \{1/n : n \in \mathbb{N}\}$ is *not* compact, without using the Heine-Borel theorem. (That is, describe an open cover that has no finite subcover.)

Since each $1/n$ is isolated, consider any U_n s.t. $U_n \cap A = \{1/n\}$.

Clearly, $\{U_n : n \in \mathbb{N}\}$ covers A and has no finite subcover. Thus A is not compact. $\square$

4. (25 points) Let X be a metric space, and $\{x_n\}$ a sequence.

   (a) (3 points each) Complete the following definitions.

      (i) The sequence $\{x_n\}$ *converges* to $x \in X$ if ...

      $\forall \varepsilon > 0, \ \exists N$ s.t. $n \geq N \Rightarrow d(x_n, x) < \varepsilon.$

      (ii) The sequence $\{x_n\}$ is a *Cauchy sequence* if ...

      $\forall \varepsilon > 0, \ \exists N$ s.t. $n, m \geq N \Rightarrow d(x_n, x_m) < \varepsilon.$

   (b) (12 points) One of these two conditions implies the other, but *not* vice-versa. State and prove the correct implication. For the false implication, give an example to show how it can fail.

(i) $\Rightarrow$ (ii)

Proof: Suppose $x_n \to x$, and fix $\varepsilon > 0$.

We know $\exists N$ s.t. $d(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq N$.

For this N, if $n, m \geq N$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\{x_n\}$ is Cauchy. $\square$

Note that (ii) $\not\Rightarrow$ (i).

Example: $\{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots\}$ is Cauchy, but does not converge in $X = \mathbb{Q}$.

(c) (2 points) Define what it means for a metric space X to be *complete.*

X is complete if every Cauchy sequence converges

(d) (1 point each) Which of the following metric spaces are complete? (No proof needed.)

(i) $\mathbb{Q}$; Not complete (See Part (b) for an example.)

(ii) $\mathbb{R}$; Complete

(iii) $\mathbb{C}$; Complete

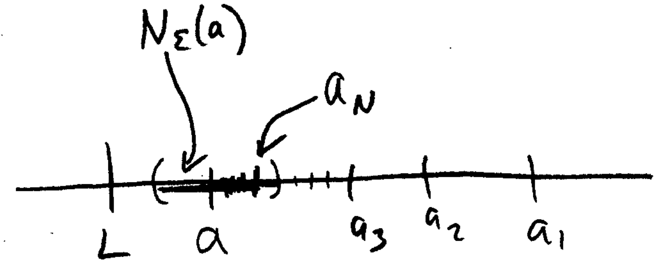
(iv) $\mathbb{R}$, with the *discrete metric*;

Complete

(v) A finite metric space X.

Complete

5. (10 points) Let $\{a_n\}$ be a monotonically decreasing sequence bounded below. That is, $a_1 \geq a_2 \geq a_3 \geq \cdots \geq L$, for some $L \in \mathbb{R}$. Prove that a_n converges. [*Hint*: First, make a conjecture about what the limit is.]



Claim: $\{a_n\}$ converges to $\boxed{a := \inf a_n}$

Proof: Fix $\varepsilon > 0$.

Since $a = \inf a_n$, $\exists$ some a_N s.t. $|a_N - a| < \varepsilon$.

Since $\{a_n\}$ is monotonically decr, if $n \geq N$,

then

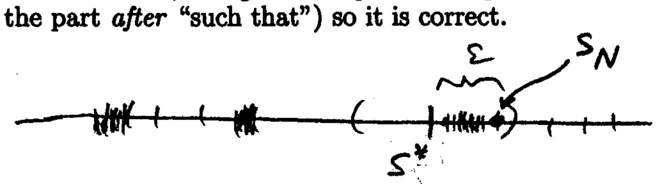
$$|a - a_n| = |a - a_n| \leq |a - a_N| < \varepsilon.$$

Thus $a_n \to a$. $\square$

6. (15 points) Suppose you have a friend taking MthSc 453 who is trying to understand the concept of limit supremum. Your friend enthuiastically writes down the following statement about a sequence $\{s_n\}$, and $s^* := \limsup s_n$:

For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|s^* - s_n| < \epsilon$.

(a) Explain in simple terms what is wrong with your friend's claim, and give an explicit example of where it fails. Then, re-write it (only modifying the part *after* "such that") so it is correct.



Problem: The above statement is saying that $\{s_n\}$ converges to s^*. This need not be the case — $\{s_n\}$ could have other smaller subsequential limits.

Correction: $\forall \varepsilon > 0$, $\exists$ $N \in \mathbb{N}$ s.t. if $n \geq N$, then $\boxed{s_n - s^* < \varepsilon}$

(b) There is actually a special case of when the statement above holds. What additional assumption about $\{s_n\}$ is needed for this to happen?

$\{s_n\}$ converges.