1. The population of a certain planet is believed to be growing according to the logistic equation. The maximum population the planet can hold is 10^{10}. In year zero the population is 50% of this maximum, and the rate of increase of the population is 10^9 per year.

   (a) What is the logistic equation satisfied by the population, $y(t)$?

   (b) How many years until the population reaches 90% of the maximum?

   (c) Sketch this solution curve in the ty-plane, as well as the steady-state solutions $y(t) = 0$ and $y(t) = 10^{10}$.

2. Consider two tanks, labeled tank A and tank B for reference. Tank A contains 100 gal of solution in which is dissolved 20 lb of salt. Tank B contains 200 gal of solution in which is dissolved 40 lb of salt. Pure water flows into tank A at a rate of 5 gal/sec. There is a drain at the bottom of tank A, and the solution leaves tank A via this drain at a rate of 5 gal/sec and flows immediately into tank B at the same rate. A drain at the bottom of tank B allows the solution to leave tank B at a rate of 2.5 gal/sec. What is the salt content in tank B at the precise moment that tank B contains 250 gal of solution.

3. For each of the second-order differential equations below, decide whether the equation is linear or nonlinear. If the equation is linear, state whether the equation is homogeneous or inhomogeneous.

   (a) $y'' + 3y' + 5y = 3\cos 2t$

   (b) $t^2y'' = 4y' - \sin t$

   (c) $t^2y'' + (1-y)y' = \cos t$

   (d) $ty'' + (\sin t)y' = 4y - \cos 5t$

   (e) $t^2y'' + 4yy' = 0$

   (f) $y'' + 4y' + 7y = 3e^{-t}\sin t$

   (g) $y'' + 3y' + 4\sin y = 0$

   (h) $(1-t^2)y'' = 3y$

4. Find the general solution to the following 2^{nd} order linear homogeneous ODEs.

   (a) $y'' + 5y' + 6y = 0$

   (b) $y'' + y' - 12y = 0$

   (c) $y'' + 4y' + 5y = 0$

   (d) $y'' + 2y = 0$

   (e) $y'' - 4y' + 4y = 0$

   (f) $4y'' + 12y' + 9y = 0$

5. In this problem, we will find all solutions to the *boundary value problem* (BVP) $y'' = \lambda y$, $y(0) = y(\pi) = 0$, where λ is a constant. This equation will turn up later when we study PDEs.

   (a) First, suppose that $\lambda = 0$. That is, solve $y'' = 0$, $y(0) = y(\pi) = 0$.

   (b) Next, suppose $\lambda = \omega^2 \geq 0$.

      (i) Solve the boundary value problem $y'' = \omega^2 y$, $y(0) = y(\pi) = 0$.

      (ii) Let $u_1(t) = \cosh \omega t = \frac{e^{\omega t}+e^{-\omega t}}{2}$ and $u_2(t) = \sinh \omega t = \frac{e^{\omega t}-e^{-\omega t}}{2}$. Show that $u_1(t)$ and $u_2(t)$ both solve $y'' = \omega^2 y$, and use this to write the general solution of this differential equation.

      (iii) Solve the boundary value problem from part (i) again, but this time, start by using the general solution you found in Part (ii) (instead of exponentials).

   (c) Finally, suppose $\lambda = -\omega^2 < 0$. That is, solve $y'' = -\omega^2 y$, $y(0) = y(\pi) = 0$.

   (d) Using your results from parts (a)–(c), describe all solutions to the boundary value problem $y'' = \lambda y$, $y(0) = y(\pi) = 0$. What are the possibile values for λ?

6. Sove the following initial value problems.

   (a) $y'' - y' - 2y = 0, \quad y(0) = -1, \quad y'(0) = 2$

   (b) $y'' - 4y' - 5y = 0, \quad y(1) = -1, \quad y'(1) = -1$

   (c) $y'' + 25y = 3, \quad y(0) = 1, \quad y'(0) = -1$

   (d) $y'' - 2y' + 17y = 0, \quad y(0) = -2, \quad y'(0) = 3$

7. Solve the following differential equations using the method of undetermined coefficients. That is, solve the related homogeneous equation and then look for a particular solution.

   (a) $y'' + y' - 12y = 24$

   (b) $y'' = -4y + 3$

   (c) $y'' + 3y' + 2y = -3e^{-4t}$

   (d) $y'' + 2y' + 2y = 2\cos 2t$

   (e) $y'' + 4y' + 4y = 4 - t$

   (f) $y'' - 2y' + y = t^2$

   (g) $y' - 2y = e^{2t}$

   (h) $y' - 3y' + 2y = e^{2t}$.

8. As we've seen, to solve ODE of the form

$$y'' + py' + qy = 0\,, \qquad p \text{ and } q \text{ constants}$$

   we assume that the solution has the form e^{rt}, and then we plug this back into the ODE to get the *characteristic equation*: $r^2 + pr + q = 0$. Given that this equation has a double root $r = r_1$ (i.e., the roots are $r_1 = r_2$), show by direct substitution that $y = te^{rt}$ is a solution of the ODE, and then write down the general solution.

9. Suppose that $z(t) = x(t) + iy(t)$ is a solution of

$$z'' + pz' + qz = Ae^{i\omega t}\,.$$

Substitute $z(t)$ into the above equation. Then compare (equate) the real and imaginary parts of each side to prove two facts:

$$\begin{aligned} x'' + px' + qx &= A\cos\omega t \\ y'' + py' + qy &= A\sin\omega t\,. \end{aligned}$$

Write a sentence or two summarizing the significance of this result.

10. If $y_f(t)$ is a solution of

$$y'' + py' + qy = f(t)$$

and $y_g(t)$ is a solution of

$$y'' + py' + qy = g(t)\,,$$

show that $z(t) = \alpha y_f(t) + \beta y_g(t)$ is a solution of

$$y'' + py' + qy = \alpha f(t) + \beta g(t)\,,$$

where α and β are any real numbers, by plugging it into the ODE.

11. Find the general solution to the following 2^{nd} order linear inhomogeneous ODEs.

(a) $y'' + 2y' + 2y = 2 + \cos 2t$

(b) $y'' + 25y = 2 + 3t + 4\cos 2t$

(c) $y'' - y = t - e^{-t}$.