1. For each part, find the determinant, eigenvalues and eigenvectors of the given matrix. If the matrix is invertible, find its inverse.

   (a) $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$ (c) $\mathbf{A} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

   (d) $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$ (e) $\mathbf{A} = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}$ (f) $\mathbf{A} = \begin{bmatrix} 5/4 & 3/4 \\ -3/4 & -1/4 \end{bmatrix}$

2. Let A be a 2×2 matrix. In this problem we will show that $\lambda = 0$ is an eigenvalue of a matrix $\mathbf{A}$ if and only if $\det(\mathbf{A}) = 0$.

   (a) Write the characteristic polynomial (i.e., the polynomial $\det(\mathbf{A} - \lambda I) = 0$) in terms of the determinant and trace of A.

   (b) Show that if $\lambda = 0$ is an eigenvalue of $\mathbf{A}$, then $\det(\mathbf{A}) = 0$.

   (c) Show that if $\det(\mathbf{A}) = 0$, then $\lambda = 0$ is an eigenvalue of $\mathbf{A}$.

   (d) Now, make the same argument – that $\lambda = 0$ is an eigenvalue if and only if $\det(\mathbf{A}) = 0$, without reference to the characteristic polynomial. (*Hint*: If $\lambda = 0$ is an eigenvalue, then $\mathbf{A}\mathbf{v} = 0\mathbf{v} = \mathbf{0}$ for some $\mathbf{v} \neq \mathbf{0}$. When does such a homogeneous system have a non-zero solution?)

3. Write each system of differential equations in matrix form, i.e., $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$. State whether the given system is autonomous or nonautonomous, and also whether it is homogeneous or inhomogeneous.

   (a) $x' = y, \quad y' = x + 4$

   (b) $x' = x + 2y + \sin t, \quad y' = -x + y - \cos t$

   (c) $x' = -2tx + y \quad y' = 3x - y$

   (d) $x' = x + 2y + 4, \quad y' = -2x + y - 3$

   (e) $x' = 3x - y, \quad y' = x + 2y$

   (f) $x' = -x + ty, \quad y' = tx - y$

   (g) $x' = x + y + 4, \quad y' = -2x + (\sin t)y$

   (h) $x' = 3x - 4y, \quad y' = x + 3y$

4. Transform the given 2^{nd} order initial value problem into an initial value problem of two 1^{st} order equations (by letting $x_1 = u$ and $x_2 = u'$), and write it in matrix form: $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}, \quad \mathbf{x}(t_0) = \mathbf{x}_0$. (No need to solve!)

   (a) $u'' + 0.25u' + 4u = 2\cos 3t, \qquad u(0) = 1, \quad u'(0) = -2$

   (b) $tu'' + u' + tu = 0, \qquad u(1) = 1, \quad u'(1) = 0$

5. In each problem below, an inhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ of two first order ODEs is given. The qualitative behavior of the solutions (as seen through their phase planes) near the critical point are all different. The motivation of this problem is to discover how the eigenvalues and eigenvectors determine the behavior of the solutions. For (a)–(e), carry out the following steps:

   (a) Find the equilibrium solution, or critical point, for the given system.

   (b) Write the associated homogeneous equation, $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and find the eigenvalues and eigenvectors of $\mathbf{A}$.

   (c) Draw a phase portrait centered at the critical point. (The PPLANE applet at `http://math.rice.edu/~dfield/dfpp.html` is fantastic.)

   (d) Describe how solutions of the system behave in the vicinity of the critical point (e.g., do they approach the critical point, depart from it, spiral around it, or something else).

   (a) $x' = -x - 4y - 4, \quad y' = x - y - 6$

   (b) $x' = -0.25x - 0.75y + 8, \quad y' = 0.5x + y - 11.5$

   (c) $x' = -2x + y - 11, \quad y' = -5x + 4y - 35$

   (d) $x' = x + y - 3, \quad y' = -x + y + 1$

   (e) $x' = -5x + 4y - 35, \quad y' = -2x + y - 11$

6. Tank A contains 10 gallons of a solution in which 5 oz of salt are dissolved. Tank B contains 20 gallons of a solution in which 6 oz of salt are dissolved. Salt water with a concentration of 2 oz/gal flows into each tank at a rate of 4 gal/min. The fully mixed solution drains from Tank A at a rate of 3 gal/min and from Tank B at a rate of 5 gal/min. Solution flows from Tank A to Tank B at a rate of 1 gal/min. Let $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, where $x_1(t)$ (respectively, $x_2(t)$) is the amount of salt in Tank A (resp., Tank B) after time t.

   (a) Write down a system of ODEs (including the initial condition $\mathbf{x}(0)$) that models this situation, and write it in matrix form: $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}, \quad \mathbf{x}(0) = \mathbf{c}$.

   (b) What is the steady-state solution, $\mathbf{x}_{ss}$?

   (c) Write down the related homogeneous equation and solve it.

   (d) Find the general solution the orginal system of differential equations modeling the tanks.

   (e) Plug in $t = 0$ and find the particular solution.

7. Suppose that $\mathbf{x}_1(t) = \mathbf{u}(t) + i\mathbf{w}(t)$ solves the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{u}$ and $\mathbf{w}$ are real-valued vector functions. Show that $\mathbf{u}$ and $\mathbf{w}$ also solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Explain your reasoning.

8. Find the general solution for each of the given system of equations. Draw a phase portrait. Describe the behavior of the solutions as $t \to \infty$.

   (a) $\mathbf{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \mathbf{x}$ (b) $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$

   (c) $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x}$ (d) $\mathbf{x}' = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \mathbf{x}$

9. In each of the next four problems, the eigenvalues and eigenvectors of a matrix A are given. Consider the corresponding system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Without using a computer, draw each of the following graphs.

   (i) Sketch a phase portrait of the system.

   (ii) Sketch the solution curve passing through the initial point $(2, 3)$.

   (iii) For the curve in part (ii), sketch the component plots of x_1 versus t and x_2 versus t on the same set of axes.

   (a) $\lambda_1 = -1, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix};$ $\lambda_2 = -4, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

   (b) $\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix};$ $\lambda_2 = -4, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

   (c) $\lambda_1 = -1, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix};$ $\lambda_2 = 4, \quad \mathbf{v}_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

   (d) $\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix};$ $\lambda_2 = 4, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$

10. In each of the next four problems, the eigenvalues and eigenvectors of a matrix A are given. Consider the corresponding system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Without using a computer, draw each of the following graphs.

    (i) Sketch a phase portrait of the system.

    (ii) Sketch the trajectory passing through the initial point $(2, 3)$.

    (a) $\lambda_1 = -4, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix};$ $\lambda_2 = -1, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

    (b) $\lambda_1 = 4, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix};$ $\lambda_2 = -1, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

    (c) $\lambda_1 = -4, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix};$ $\lambda_2 = 1, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

    (d) $\lambda_1 = 4, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix};$ $\lambda_2 = 1, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$

11. Find the general solution for each of the given systems in terms of *real-valued functions*, and draw a phase portrait. Describe the behavior of the solutions as $t \to \infty$.

    (a) $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \mathbf{x}$ (b) $\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \mathbf{x}$ (c) $\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} \mathbf{x}$