1. Use the Heaviside function to concisely write each piecewise function.

(a) $f(t) = \begin{cases} 5 & 2 \leq t < 4; \\ 0 & \text{otherwise} \end{cases}$

(b) $f(t) = \begin{cases} 0 & t < 0; \\ t & 0 \leq t < 3 \\ 4 & t \geq 3 \end{cases}$

(c) $f(t) = \begin{cases} 0 & t < 0; \\ t^2 & 0 \leq t < 2 \\ 4 & t \geq 2 \end{cases}$

2. Find the inverse Laplace transform of each function. Create a piecewise definition for your solution that does not use the Heavyside function.

(a) $F(s) = \dfrac{e^{-2s}}{s+3}$ (b) $F(s) = \dfrac{1-e^{-s}}{s^2}$ (c) $F(s) = \dfrac{e^{-s}}{s^2+4}$

3. For each initial value problem, sketch the forcing term, and then solve for $y(t)$. Write your solution as a piecewise function (i.e., not using the Heavysie function). Recall that the function $H_{ab}(t) = H(t-a) - H(t-b)$ is the interval function.

(a) $y'' + 4y = H_{01}(t), \quad y(0) = 0, \ y'(0) = 0$

(b) $y'' + 4y = t\, H_{01}(t), \quad y(0) = 0, \ y'(0) = 0$

4. Define the function $\delta_p^\epsilon(t) = \frac{1}{\epsilon}\left(H_p(t) - H_{p+\epsilon}(t)\right)$.

(a) Sketch $\delta_p^\epsilon(t)$, and show that its Laplace $\delta_p^\epsilon(t)$ is

$$\mathcal{L}\left\{\delta_p^\epsilon(t)\right\} = e^{-sp}\,\frac{1-e^{-s\epsilon}}{s\epsilon}\,.$$

(b) Use l'Hôpital's rule to take the limit of the result in part (a) as $\epsilon \to 0$. How does this result agree with the fact that $\mathcal{L}\{\delta_p(t)\} = e^{-sp}$?

5. Engineers likes to that the "derivative of a unit step is a unit impulse". In this problem you will further explore this idea. Define the function

$$H_p^\epsilon(t) = \begin{cases} 0, & 0 \leq t < p \\ \frac{1}{\epsilon}(t-p), & p \leq t < p+\epsilon \\ 1, & t \geq p+\epsilon \end{cases}$$

(a) Sketch the graph of $H_p^\epsilon(t)$.

(b) Without being too precise about things, we could argue that $H_p^\epsilon(t) \to H_p(t)$ as $\epsilon \to 0$, where $H_p(t) = H(t-p)$. Sketch the graph of the derivative of $H_p^\epsilon(t)$.

(c) Compare the graph of $\delta_p^\epsilon(t)$ to Part (b). Argue (graphically) that $H_p'(t) = \delta_p(t)$.

(d) Use a Laplace transform to solve the following initial value problem:

$$y' = \delta_p(t), \qquad y(0) = 0\,.$$

Why does this also suggest (but this time, algebraically) that $H_p'(t) = \delta_p(t)$?

6. Solve the following initial value problems.

   (a) $y'' + 4y = \delta(t), \quad y(0) = 0, \;\; y'(0) = 0$

   (b) $y'' - 4y' - 5y = \delta(t), \quad y(0) = 0, \;\; y'(0) = 0$

7. Compute the following convolutions.

   (a) $e^{at} * e^{bt}, \quad a \neq b$ (b) $e^{at} * e^{at}$, (c) $t * e^{t}$.

8. Compute the following inverse Laplace transforms *without* using partial fraction decomposition. Use convolutions for Parts (b)–(d) instead.

   (a) $\mathcal{L}^{-1}\left(\dfrac{3s-5}{s-1}\right)$

   (b) $\mathcal{L}^{-1}\left(\dfrac{1}{s(s+1)}\right)$

   (c) $\mathcal{L}^{-1}\left(\dfrac{s}{(s-1)(s^2+1)}\right)$

   (d) $\mathcal{L}^{-1}\left(\dfrac{s^2}{(s^2+9)^2}\right)$.

9. Find the Fourier series of the following functions *without* computing any integrals.

   (a) $f(x) = 2 - 3\sin 4x + 5\cos 6x$,

   (b) $f(x) = \sin^2 x$ [*Hint*: Use a standard trig identity.]

10. The function
$$f(x) = \begin{cases} 0 & -\pi \leq x < -\pi/2 \\ 1 & -\pi/2 \leq x < \pi/2 \\ 0 & \pi/2 \leq x \leq \pi \end{cases}$$
can be extended to be periodic of period 2π. Sketch the graph of the resulting function on $[-5\pi, 5\pi]$ and compute its Fourier series.

11. The function
$$f(x) = \begin{cases} 0 & -\pi \leq x < 0 \\ x & 0 \leq x \leq \pi \end{cases}$$
can be extended to be periodic of period 2π. Sketch the graph of the resulting function on $[-5\pi, 5\pi]$ and compute its Fourier series.