1. Consider the 2π-periodic function defined by

$$f(x) = \begin{cases} x^2 & -\pi \leq x < \pi, \\ f(x-2k\pi), & -\pi + 2k\pi \leq x < \pi + 2k\pi. \end{cases}$$

   (a) Sketch the graph of $f(x)$ on $[-5\pi, 5\pi]$.

   (b) Compute the Fourier series of $f(x)$.

   (c) Solve the differential equation $x''(t) + \omega_0^2 x(t) = f(t)$. Look for a particular solution of the form

$$x_p(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt\,.$$

   [*Hint*: Can you deduce right away that some of the coefficients will be zero?]

2. Determine which of the following functions are even, which are odd, and which are neither:

   (a) $f(x) = x^3 + 3x$

   (b) $f(x) = x^2 + |x|$

   (c) $f(x) = 1/x$

   (d) $f(x) = e^x$

   (e) $f(x) = \frac{1}{2}(e^x + e^{-x})$

   (f) $f(x) = \frac{1}{2}(e^x - e^{-x})$.

3. Suppose that f is a function defined on $\mathbb{R}$ (not necessarily periodic). Show that there is an odd function f_{odd} and an even function f_{even} such that $f(x) = f_{\text{odd}} + f_{\text{even}}$. [*Hint*: As a guiding example, suppose $f(x) = e^{ix}$, and consider $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ and $i \sin x = \frac{1}{2}(e^{ix} - e^{-ix})$.]

4. Express the y-intercept of $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ in terms of the a_n's and b_n's. (*Hint*: It's *not* a_0 or $a_0/2$!)

5. Consider the 2π-periodic function $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$. Write the Fourier series for the following functions:

   (a) The reflection of $f(x)$ across the y-axis;

   (b) The reflection of $f(x)$ across the x-axis;

   (c) The reflection of $f(x)$ across the origin.

6. Consider the function defined on the interval $[0, \pi]$:

$$f(x) = x(\pi - x).$$

   (a) Sketch the even extension of this function and find its Fourier cosine series.

   (b) Sketch the odd extension of this function and find its Fourier sine series.

7. (a) The Fourier series of an odd function consists only of sine-terms. What additional symmetry conditions on f will imply that the sine coefficients with even indices will be zero (i.e., each $b_{2n} = 0$)? Give an example of a non-zero function satisfying this additional condition.

   (b) What symmetry conditions on f will imply that the sine coefficients with odd indices will be zero (i.e., each $b_{2n+1} = 0$)? Give an example of a non-zero function satisfying this additional condition.

   (c) Sketch the graph of a non-zero even function, such that $a_{2n} = 0$ for all n.

   (d) Sketch the graph of a non-zero even function, such that $a_{2n+1} = 0$ for all n.

8. Compute the complex Fourier series for the function defined on the interval $[-\pi, \pi]$:

$$f(x) = \begin{cases} -1, & -\pi \le x < 0, \\ 5, & 0 \le x \le \pi. \end{cases}$$

   Use the c_n's to find the coefficients of the real Fourier series [*Hint: Use* $a_n = c_n + c_{-n}$, *and* $b_n = i(c_n - c_{-n})$.]

9. On a previous problem, you derived the real Fourier series for the function defined by

$$f(x) = x^2 \qquad \text{for } -\pi < x \le \pi\,.$$

   and extended to be periodic of period 2π.

   (a) Compute the complex Fourier coefficients from the real coefficients using the identities $c_n = (a_n - ib_n)/2$ and $c_{-n} = (a_n + ib_n)/2$.

   (b) Use the real form of the Fourier series and *Parseval's identity* to compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.

10. Compute $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$. *Hint*: Compute the Fourier series for $f(x) = |x|$, and then observe that $f(\pi) = \pi$. (Parseval's identity not needed!)