1. Let $u(x,t)$ be the temperature of a bar of length 10 that is fully insulated so that no heat can enter or leave. Suppose that initially, the temperature is increasing linearly from 70° at one endpoint to 80° at the other endpoint.
   (a) Sketch the initial heat distribution on the bar and express it as a function of x.
   (b) Write down an initial/boundary value problem to which $u(x,t)$ is a solution. (Let the constant from the heat equation be c^2.)
   (c) What will the steady-state solution be?

2. Let $u(x,t)$ be the temperature of a bar of length 10 that is fully insulated at its right endpoint but uninsulated at its left endpoint. Suppose the bar is sitting in a 70° room and that initially, the temperature of the bar increases linearly from 70° at the left endpoint to 80° at the other end. Finally, suppose the interior of the bar is poorly insulated, so heat can escape.
   (a) Suppose that heat escapes at a constant rate of 1° per hour. Write an initial/boundary value problem for $u(x,t)$ that could model this situation.
   (b) A more realistic situation would be for heat to escape not at a constant rate, but at a rate proportional to the *difference* between the temperature of the bar and the ambient temperature of the room. Write an initial/boundary value problem for $u(x,t)$ that could model this situation. What is the steady-state solution and why?

3. Consider the following PDE:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = 0, \quad u_x(\pi,t) + \gamma\, u(\pi,t) = 0, \qquad u(x,0) = h(x)\ ,$$

   where γ is a constant, and $h(x)$ and arbitrary function on $[0,\pi]$.
   (a) Describe a physical situation that this models. Be sure to describe the impact of the initial condition, both boundary conditions, and the constant γ.
   (b) What is the steady-state solution and why? (Use your physical intuition).

4. In this problem, we will find all solutions to the *boundary value problem* (BVP) $y'' = \lambda y$, $y'(0) = y'(\pi) = 0$, where λ is a constant. These are called *Neumann* boundary conditions.
   (a) First, suppose that $\lambda = 0$. That is, solve $y'' = 0$; $y'(0) = y'(\pi) = 0$.
   (b) Next, suppose $\lambda = \omega^2 \geq 0$. That is, solve the boundary value problem $y'' = \omega^2 y$; $y'(0) = y'(\pi) = 0$. Use hyperbolic sine and cosine functions for your general solution instead of exponentials. (It will make things easier!)
   (c) Finally, suppose $\lambda = -\omega^2 < 0$. That is, solve $y'' = -\omega^2 y$; $y'(0) = y'(\pi) = 0$.
   (d) Using your results from parts (a)–(c), describe all solutions to the boundary value problem $y'' = \lambda y$; $y'(0) = y'(\pi) = 0$. What are the possibile values for λ?

5. Solve the boundary value problem that has *mixed boundary conditions*:

$$y'' = \lambda y, \qquad y(0) = y'(\pi) = 0\,.$$

That is, find all possible values of λ that lead to a nonzero solution, and find those solutions.

6. We will solve for the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the following conditions:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = 5\sin x + 3\sin 2x.$$

(a) Briefly describe and sketch a physical situation which this models. Be sure to explain the effect of both boundary conditions (called *Dirichlet* boundary conditions) and the initial condition.

(b) Assume that $u(x,t) = f(x)g(t)$. Find u_t and u_{xx}. Also, determine the boundary conditions for $f(x)$ (at $x = 0$ and $x = \pi$) from the boundary conditions for $u(x,t)$.

(c) Plug $u = fg$ back into the PDE and separate variables by dividing both sides of the equation by $c^2 fg$. Set this equal to a constant λ and write down two ODEs: one for $g(t)$ and a BVP for $f(x)$.

(d) Recall from HW 3 that the BVP for f has a solution $f_n(x)$ for each $\lambda = -n^2$ where $n = 1, 2, \ldots$, and that solution is $f_n(x) = b_n \sin nx$. Now, given such $\lambda = -n^2$, solve the ODE for g. Call this solution $g_n(t)$.

(e) Using your solution to Part (d) and the principle of superposition, find the general solution to the PDE.

(f) Solve the remaining *initial value problem*, i.e., find the particular solution $u(x,t)$ that additionally satisfies $u(x,0) = 5\sin x + 3\sin 2x$. [Your solution should be a sum of only two terms – and *not* have a $\sum$ in it!]

(g) What is the steady-state solution, i.e., $u_{ss}(x) := \lim_{t\to\infty} u(x,t)$?

7. Consider a similar situation as the previous problem but with *inhomogeneous* boundary conditions.

$$u_t = c^2 u_{xx}, \qquad u(0,t) = 30, \quad u(\pi,t) = 100, \qquad u(x,0) = 30 + \tfrac{70}{\pi}x + 5\sin x + 3\sin 2x.$$

(a) Describe and sketch a physical situation that this models. Be sure to describe the impact of *both* boundary conditions and the initial condition.

(b) Use your physical intuition to determine what the steady-state solution $u_{ss}(x)$ is.

(c) Define $v(x,t) = u(x,t) - u_{ss}(x)$, where $u_{ss}(x)$ is your solution to Part (b). Rewrite the PDE and the initial and boundary conditions in terms of v instead of u. The resulting PDE is *homogeneous* because $v(x,t) = 0$ is a solution.

(d) Write down the solution to this PDE by adding the steady-state solution to the solution of the related homogeneous problem (which you've already solved!).

8. Consider the following initial/boundary value problem for the heat equation:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = 0, \quad u_x(\pi,t) = 0, \qquad u(x,0) = 3\sin\tfrac{5x}{2}.$$

   (a) Describe (and sketch) a physical situation that this models. Be sure to describe the impact of *both* boundary conditions and the initial condition.

   (b) Assume there is a solution of the form $u(x,t) = f(x)g(t)$. Find u_t, u_x, and u_{xx}. Also, determine the boundary conditions for $f(x)$ (at $x = 0$ and $x = \pi$) from the *mixed boundary conditions* for $u(x,t)$.

   (c) Plug $u = fg$ back into the PDE and separate variables by dividing both sides of the equation by $c^2 fg$. Set this equal to a constant λ and write down two ODEs: one for $g(t)$, and a BVP for $f(x)$.

   (d) Write down the solution to the BVP for f (see Problem 5) and to the ODE for g. There should be one for each $n = 0, 1, 2, \dots$.

   (e) Write down the general solution to the PDE.

   (f) Find the particular solution $u(x,t)$ that additionally satisfies the initial condition $u(x,0) = 3\sin(5x/2)$. [Again, it should not contain a $\sum$!]

   (g) What is the steady-state solution?

9. Consider the heat equation with *periodic boundary conditions*:

$$u_t = c^2 u_{\theta\theta}, \qquad u(\theta + 2\pi, t) = u(\theta, t), \qquad u(\theta, 0) = 2 + 4\sin 3\theta - \cos 5\theta.$$

   (a) Describe and sketch a situation that this models.

   (b) Assume there is a solution of the form $u(\theta,t) = f(\theta)g(t)$. Find u_t and $u_{\theta\theta}$. Use the periodic boundary conditions for $u(\theta,t)$ to derive similar periodic boundary conditions for $f(\theta)$.

   (c) Plug $u = fg$ back into the PDE and separate variables by dividing both sides of the equation by $c^2 fg$. Set this equal to a constant λ and write down two ODEs: one for $g(t)$, and a "periodic" BVP for $f(\theta)$.

   (d) Solve for $g(t)$, $f(\theta)$, and λ. [*Note*: You won't be able to conclude that $a = 0$ or $b = 0$ – so unlike before, they'll both stick around.]

   (e) Find the general solution of the boundary value problem. As before, it will be a superposition (infinite sum) of solutions $u_n(\theta,t) = f_n(\theta)g_n(t)$.

   (f) Find the particular solution to the initial value problem that satisfies the initial condition $u(\theta,0) = 2 + 4\sin 3\theta - \cos 5\theta$.

   (g) What is the steady-state solution? Give a mathematical *and* intuitive (physical) justification for this.