1. Consider the function $u(x,t)$ defined for $0 \leq x \leq \pi$ and $t \geq 0$ which satisfies the following initial/boundary value problem of the *wave equation*:

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = u(\pi,t) = 0, \qquad \begin{array}{l} u(x,0) = x(\pi - x), \\ u_t(x,0) = 0. \end{array}$$

   (a) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of both boundary conditions and both initial conditions.

   (b) Assume that $u(x,t) = f(x)g(t)$. Find u_t, u_{tt}, and u_{xx}. Also, determine *two* boundary conditions for $f(x)$ (at $x = 0$ and $x = \pi$) from the boundary conditions for $u(x,t)$, and *one* initial condition for $g(t)$.

   (c) Plug $u = fg$ back into the PDE and separate variables by dividing both sides of the equation by $c^2 fg$. Set this equal to a constant λ and write down two ODEs: one for $g(t)$ and a BVP for $f(x)$.

   (d) Recall from HW 3 that the BVP for f has a solution $f_n(x)$ for each $\lambda = -n^2$ where $n = 1, 2, \dots$, and that solution is $f_n(x) = b_n \sin nx$. Now, given such $\lambda = -n^2$, solve the ODE for g. Call this solution $g_n(t)$. Use the one initial condition for g to force one of the constants to be zero.

   (e) Using your solution to Part (d) and the principle of superposition, find the general solution to the PDE.

   (f) Solve the remaining *initial value problem*, i.e., find the particular solution $u(x,t)$ that additionally satisfies $u(x,0) = x(\pi - x)$.

   (g) What is the long-term behavior of this solution (i.e., what happens as $t \to \infty$)?

2. Consider the function $u(x,t)$ defined for $0 \leq x \leq \pi$ and $t \geq 0$ which satisfies different initial conditions:

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = u(\pi,t) = 0, \qquad \begin{array}{l} u(x,0) = 0, \\ u_t(x,0) = x(\pi - x). \end{array}$$

   Describe the difference in the physical situation that this models to that of the previous problem, and then solve it. Note that many of the steps will be identical – *you do not need to re-derive the solutions of anything you have previously solved!*

3. For each of the following functions, compute the Laplacian $\nabla^2 f$ and determine if it is harmonic.

   (a) $f(x) = 10 - 3x$.

   (b) $f(x,y) = x^2 + y^2$.

   (c) $f(x,y) = x^2 - y^2$.

   (d) $f(x,y) = e^x \cos y$.

   (e) $f(x,y) = x^3 - 3xy^2$.

   (f) $f(x,y) = \ln(x^2 + y^2)$.

4. (a) Solve the following Dirichlet problem for Laplace's equation in a square region: Find $u(x, y)$, $0 \le x \le \pi$, $0 \le y \le \pi$ such that

$$\nabla^2 u = 0, \qquad \begin{aligned} &u(0, y) = u(\pi, y) = 0, \\ &u(x, 0) = 0, \quad u(x, \pi) = x(\pi - x). \end{aligned}$$

   (b) Solve the following Dirichlet problem for Laplace's equation in the same square region: Find $u(x, y)$, $0 \le x \le \pi$, $0 \le y \le \pi$ such that

$$\nabla^2 u = 0, \qquad \begin{aligned} &u(0, y) = 0, \quad u(\pi, y) = 4 \sin y - 3 \sin 2y + 2 \sin 3y, \\ &u(x, 0) = u(x, \pi) = 0. \end{aligned}$$

   (c) By adding the solutions to parts (a) and (b) together (superposition), find the solution to the Dirichlet problem: Find $u(x, y)$, $0 \le x \le \pi$, $0 \le y \le \pi$ such that

$$\nabla^2 u = 0, \qquad \begin{aligned} &u(0, y) = 0, \quad u(\pi, y) = 4 \sin y - 3 \sin 2y + 2 \sin 3y, \\ &u(x, 0) = 0, \quad u(x, \pi) = x(\pi - x). \end{aligned}$$

   (d) Sketch the solutions to (a), (b), and (c). *Hint: it is enough to sketch the boundaries, and then use the fact that the solutions are harmonic functions.*

   (e) Consider the heat equation in a square region, along with the following boundary conditions:

$$u_t = c^2 \nabla^2 u, \qquad \begin{aligned} &u(0, y) = 0, \quad u(\pi, y) = 4 \sin y - 3 \sin 2y + 2 \sin 3y, \\ &u(x, 0) = 0, \quad u(x, \pi) = x(\pi - x). \end{aligned}$$

   What is the steady-state solution? (Note: This will *not* depend on the initial conditions!)

5. Consider the following initial/boundary value problem for the heat equation in a square region, and the function $u(x, y, t)$, where $0 \le x \le \pi$, $0 \le y \le \pi$ and $t \ge 0$.

$$u_t = c^2 \nabla^2 u, \qquad \begin{aligned} &u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0 \\ &u(x, y, 0) = 2 \sin x \sin y + 5 \sin 2x \sin y. \end{aligned}$$

   (a) Briefly describe and sketch (very roughly) a physical situation which this models. Be sure to explain the effect of the boundary conditions and the initial condition.

   (b) Assume that the solution has the form $u(x, y, t) = f(x, y) g(t)$. Find u_{xx}, u_{yy}, and u_t.

   (c) Plug $u = fg$ back into the PDE and separate variables by dividing both sides of the equation by $c^2 fg$. Set this equal to a constant λ and write down an ODE for $g(t)$ and a PDE for $f(x, y)$ (the *Helmholz equation*). Include *four* boundary conditions for $f(x, y)$.

   (d) Recall from class that for every pair (n, m) of positive integers, the Helmholz equation has a solution $f_{nm}(x, y) = b_{nm} \sin nx \sin mx$ for $\lambda_{nm} = -(n^2 + m^2)$. Now, solve the ODE for $g(t)$.

(e) Find the general solution to the boundary value problem. It will be a superposition (infinite sum) of solutions $u_{nm}(x, y, t) = f_{nm}(x, y)g_{nm}(t)$.

(f) Find the particular solution to the initial value problem that additionally satisfies the initial condition $u(x, y, 0) = 2 \sin x \sin y + 5 \sin 2x \sin y$.

(g) What is the steady-state solution? Give a mathematical *and* intuitive (physical) justification.

6. Consider the following boundary value problem for the 2D heat equation:

$$u_t = c^2\nabla^2 u, \qquad u(x, 0, t) = u(0, y, t) = u(\pi, y, t) = 0, \quad u(x, \pi, t) = x(\pi - x).$$

(a) What is the steady-state solution, $u_{ss}(x, y)$? [Hint: Look at a previous problem on Laplace's equation]. Sketch it.

(b) Define $v(x, y, t) = u(x, y, t) - u_{ss}(x, y)$, where $u_{ss}(x, y)$ is your solution to Part (a). Rewrite the PDE including the boundary conditions in terms of v instead of u. The resulting PDE is *homogeneous* because $v(x, y, t) = 0$ is a solution.

(c) Write down the *general solution* to this PDE by adding the steady-state solution to the solution of the related homogeneous problem (which you've already solved!).