MthSc 208: Differential Equations (Summer I, 2013)
In-class Worksheet 7a: The Heat Equation

NAME:

We will solve for the function $u(x,t)$ defined for $0 \leq x \leq \pi$ and $t \geq 0$ which satisfies the following initial value problem of the heat equation:

$$u_t = c^2 u_{xx} \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = x(\pi - x),$$

(a) Carefully descsribe (and sketch) a physical situation that this models.

(b) Assume that $u(x,t) = f(x)g(t)$. Compute u_t and u_{xx}, and derive boundary conditions for $f(x)$.

(c) Plug $u = fg$ back into the PDE and separate variables by dividing both sides of the equation by $c^2 fg$. Now set this equal to a constant λ, and write down two ODEs: one for $g(t)$, and a *boundary value problem* (BVP) for $f(x)$.

(d) Recall from the homework that the BVP for f has a solution $f_n(x)$ for each $\lambda = -n^2$ where $n = 1, 2, \dots,$ and that solution is $f_n(x) = b_n \sin nx$. Now, given such $\lambda = -n^2$ solve the ODE for g. Call this solution $g_n(t)$.

(e) Find the general solution of the PDE. As before, it will be a superposition (infinite sum) of solutions $u_n(x,t) = f_n(x)g_n(t)$.

(f) Find the particular solution to the original initial/boundary value problem by using the initial condition. The following information is useful:

The Fourier sine series of $x(\pi - x)$ is $\displaystyle\sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3} \sin nx$.

(g) Consider the following initial/boundary problem for heat equation.

$$u_t = c^2 u_{xx} \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = 3\sin 2x + 5\sin 6x.$$

The only difference between this problem and the previous is in the initial condition, thus it will have the same general solution. Repeat part (f) to find the particular solution.

(h) What is the steady-state solution to the function $u(x,t)$, both in part (f), and in part (g)? Give a physical interpretation for this quantity.