MthSc 208: Differential Equations (Summer I, 2013)
In-class Worksheet 7b: The Wave Equation

NAME:

We will solve for the function $u(x,t)$ defined for $0 \leq x \leq \pi$ and $t \geq 0$ which satisfies the following initial value problem of the wave equation:

$$u_{tt} = c^2 u_{xx} \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = x(\pi - x), \qquad u_t(x,0) = 1\,.$$

(a) Carefully descsribe (and sketch) a physical situation that this models.

(b) Assume that $u(x,t) = f(x)g(t)$. Compute u_t, u_{tt}, u_x, u_{xx}, and find boundary conditions for $f(x)$.

(c) Plug $u = fg$ back into the PDE and separate variables by dividing both sides of the equation by $c^2 fg$. Set this equal to a constant λ, and write down two ODEs: one for $g(t)$, and a *boundary value problem* (BVP) for $f(x)$.

(d) Recall from the homework that the BVP for f has a solution $f_n(x)$ for each $\lambda = -n^2$ where $n = 1, 2, \dots,$ and that solution is $f_n(x) = b_n \sin nx$. Now, given such $\lambda = -n^2$ solve the ODE for g. Call this solution $g_n(t)$.

(e) Find the general solution of the PDE. As before, it will be a superposition (infinite sum) of solutions $u_n(x,t) = f_n(x)g_n(t)$.

(f) Find the particular solution to the original initial/boundary value problem by using the initial conditions. The following information is useful:

The Fourier sine series of $x(\pi - x)$ is $\displaystyle\sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3} \sin nx$.

(g) What is the long-term behavior of the system? Give a mathematical, and physical, justification.