

MthSc 208: Differential Equations (Summer I, 2013)

In-class Worksheet 7c: The 2D Heat Equation

NAME:

We will solve for the function $u(x, y, t)$ defined for $0 \leq x, y \leq \pi$ and $t \geq 0$ which satisfies the following initial value problem of the heat equation:

$$\begin{aligned}u_t &= c^2(u_{xx} + u_{yy}) & u(0, y, t) = u(\pi, y, t) = u(x, 0, t) = u(x, \pi, t) &= 0, \\u(x, y, 0) &= 2 \sin x \sin 2y + 3 \sin 4x \sin 5y.\end{aligned}$$

(a) Carefully describe (and sketch) a physical situation that this models.

(b) Assume that $u(x, y, t) = f(x, y)g(t)$. Compute u_{xx} , u_{yy} , and u_t , find boundary conditions for $f(x, y)$.

- (c) Plug $u = fg$ back into the PDE and separate variables by dividing both sides of the equation by $c^2 fg$. Set this equal to a constant λ , and write down two equations: an ODE for $g(t)$, and a PDE $f(x, y)$ (called the *Helmholtz equation*), with four boundary conditions.
- (d) Solve the ODE for $g(t)$.
- (e) To solve the PDE for f , assume that $f(x, y) = X(x)Y(y)$. Plug this back in and separate variables. [For consistency, put the X''/X term on one side of the equation, and set equal to a constant μ .]

- (f) Write down two BVPs – one for $X(x)$ and one for $Y(y)$, [*Hint*: It is easier notationally if you introduce a new constant, $\nu := \lambda - \mu$.]
- (g) Solve the BVPs for $X(x)$ and $Y(y)$, and determine μ and ν (and hence λ). You should get a λ for each choice of positive integers $n, m \in \mathbb{N}$, call it λ_{nm} .
- (h) For each $n, m \in \mathbb{N}$, we have a solution $u_{nm}(x, y, t) = f_{nm}(x, y)g_{nm}(t)$. Write down this solution.

(i) Find the general solution of the PDE. It will be a doubly infinite sum (superposition) of solutions:

$$\sum_{n,m \in \mathbb{N}} u_{nm}(x, y, t).$$

(j) Find the particular solution to the initial value problem by using the initial condition.

(k) What is the long-term behavior of the system? Give a mathematical, and physical, justification.