1. Let $f \in \mathcal{B}[a,b]$, and suppose that $\mathcal{P}$ and $\mathcal{P}^*$ are partitions of $[a,b]$ with $\mathcal{P}^* \supset \mathcal{P}$. Prove that $\mathcal{U}(\mathcal{P}^*, f) \leq \mathcal{U}(\mathcal{P}, f)$.

2. The method used in Example 6.1.6 (d) can be summarized as the following theorem:

   Let $f \in \mathcal{B}[a,b]$ and suppose there exists a sequence $\{\mathcal{P}_n\}_{n=1}^{\infty}$ of partitions of $[a,b]$ such that
   $$\lim_{n\to\infty} \mathcal{L}(\mathcal{P}_n, f) = \lim_{n\to\infty} \mathcal{U}(\mathcal{P}_n, f) = L \in \mathbb{R}.$$
   Then $f \in \mathcal{R}[a,b]$ and $\int_a^b f = L$. Prove this.

3. Use the above theorem to compute $\displaystyle\int_a^b x\,dx$.

4. For each of the given functions, check the Riemann integrability and evaluate the Riemann integral on the specificed interval.
   $$\text{(a)} \quad f(x) = \begin{cases} -1, & x \in [0,1) \\ 2, & x \in [1,2] \end{cases} \quad \text{on } [0,2]. \qquad \text{(b)} \quad f(x) = 2x+1 \text{ on } [0,1]\,.$$

5. Let $f(x) = \begin{cases} 0, & x \in \mathbb{Q} \cap [0,1] \\ x, & x \in \mathbb{Q}^c \cap [0,1]. \end{cases}$ Compute $\underline{\int_0^1} f$ and $\overline{\int_0^1} f$ and determine if $f \in \mathcal{R}[0,1]$.

6. Let $f, g \in \mathcal{B}[a,b]$ and $c \in \mathbb{R}$.

   (a) Prove that $\displaystyle\underline{\int_a^b} f + \underline{\int_a^b} g \leq \underline{\int_a^b} (f+g)$, and find an example that satisfies
   $$\underline{\int_a^b} f + \underline{\int_a^b} g < \underline{\int_a^b} (f+g) = \overline{\int_a^b} (f+g) < \overline{\int_a^b} f + \overline{\int_a^b} g\,.$$

   (b) Prove $\displaystyle\overline{\int_a^b} (cf) = \begin{cases} c\overline{\int_a^b} f & \text{if } c \geq 0, \\ c\underline{\int_a^b} f & \text{if } c < 0, \end{cases} \qquad \underline{\int_a^b} (cf) = \begin{cases} c\underline{\int_a^b} f & \text{if } c \geq 0, \\ c\overline{\int_a^b} f & \text{if } c < 0, \end{cases}$

   (c) Prove that if $f \geq g$, then $\displaystyle\underline{\int_a^b} f \geq \underline{\int_a^b} g$ and $\displaystyle\overline{\int_a^b} f \geq \overline{\int_a^b} g$.

   (d) Prove $\displaystyle\left|\overline{\int_a^b} f\right| \leq \overline{\int_a^b} |f|$ and $\displaystyle\left|\underline{\int_a^b} f\right| \leq \overline{\int_a^b} |f|$. Find an example s.t. $\displaystyle\left|\underline{\int_a^b} f\right| \leq \underline{\int_a^b} |f|$.

7. Prove the following for $f \geq 0$:
   (a) If $f \in \mathcal{C}[0,1]$, then $\int_0^1 f = 0$ implies $f = 0$.
   (b) If we remove the continuity condition, then (a) need not hold.
   (c) If $f \in \mathcal{R}[0,1]$, then $f(x) = 0$ for all $x \in \mathbb{Q} \cap [0,1]$ implies $\int_0^1 f = 0$.

8. Let $0 < f \in \mathcal{R}[0,1]$. Prove or disprove the following:

   (a) $\displaystyle\int_0^1 f > 0\,,$ (b) $\displaystyle\frac{1}{f} \in \mathcal{R}[0,1]\,.$

9. Find a function $f \in \mathcal{B}[0,1]$ for which $f^2 \in \mathcal{R}[0,1]$ but $f \notin \mathcal{R}[0,1]$.