1. Let $f \in \mathcal{C}[a,b]$. Prove that if $\displaystyle\int_a^b fg = 0$ for all $g \in \mathcal{R}[a,b]$, then $f = 0$.

2. Using Lebesgue's theorem, show that $f + g \in \mathcal{R}[a,b]$ for any $f, g \in \mathcal{R}[a,b]$.

3. Let $f(x) = \begin{cases} 1, & x = 0 \\ \frac{1}{n}, & x = \frac{m}{n} \in \mathbb{Q} \cap (0,1] \\ 0, & x \in \mathbb{Q}^c \cap [0,1] \end{cases}$ Prove that f is continuous at all irrational points.

4. Prove, for $f \in \mathcal{B}[a,b]$ and a partition $\mathcal{Q} = \{y_0, y_1, \ldots, y_K\}$,
$$\mathcal{U}(\mathcal{P}, f) - (K-1)(M-m)||\mathcal{P}|| \le \mathcal{U}(\mathcal{P} \cup \mathcal{Q}, f), \quad \forall \mathcal{P}: \text{ any partition.}$$
Hint: First, draw an appropriate picture that contains the essential idea of the proof.

5. For $f \in \mathcal{B}[a,b]$, prove that $\displaystyle\overline{\int_a^b} f = \lim_{||\mathcal{P}|| \to 0} \mathcal{U}(\mathcal{P}, f)$, where $||\mathcal{P}|| = \max \Delta x_i$.

6. Let $f(t) = \begin{cases} t, & t \in [0,1) \\ b - t^2, & t \in [1,2]. \end{cases}$
   (a) Find the indefinite integral, $G(x) := \displaystyle\int_0^x f(t)\, dt$ for all $x \in [0,2]$.
   (b) For what value of b is $G(x)$ differentiable for all $x \in [0,2]$?

7. Find $F'(x)$, where F is defined on $[0,1]$ as follows:
   (a) $F(x) = \displaystyle\int_x^1 \sqrt{1+t^3}$.
   (b) $F(x) = \displaystyle\int_0^{x^2} f(t)\, dt$, where f is continuous.
   (c) $F(x) = \displaystyle\int_{h(x)}^{g(x)} f(t)\, dt$, where f is continuous and g and h are differentiable.
   (d) $F(x) = \displaystyle\int_{x-a}^{x+a} f(t)\, dt$, where f is continuous and $a > 0$.

8. Let $f \in \mathcal{C}[a,b]$ and $g \in \mathcal{R}[a,b]$ with $g \ge 0$. Prove that
$$\exists c \in [a,b] \text{ such that } \int_a^b f(x)g(x)\, dx = f(c) \int_a^b g(x)\, dx.$$

9. Let $f \in \mathcal{C}[0,1]$. Prove that $\displaystyle\lim_{n\to\infty} \int_0^1 f(x^n)\, dx = f(0)$.