1. For $n \geq 2$, define $f_n(x) = \begin{cases} n^2 x & \text{on } [0, \frac{1}{n}], \\ 2n - n^2 x & \text{on } (\frac{1}{n}, \frac{2}{n}], \\ 0 & \text{on } (\frac{2}{n}, 1]. \end{cases}$ Sketch $f_n(x)$ and show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\, dx\,.$$

2. Let $f_n(x) = \dfrac{x^n}{1+x^n}$ on $[0,1]$.
   (a) Prove that f_n converges uniformly to 0 on $[0,\epsilon]$ for all $\epsilon \in (0,1)$.
   (b) Does f_n converge uniformly on $[0,1]$? Prove or disprove.
3. Prove that if f_n converges uniformly on (a,b) and $f_n(a)$ and $f_n(b)$ converge, then f_n converges uniformly on $[a,b]$.
4. Let f be uniformly continuous on $\mathbb{R}$ and $f_n(x) := f(x + \frac{1}{n})$ for all $n \in \mathbb{N}$. Prove that f_n converges uniformly to f on $\mathbb{R}$.
5. Find an example of each and prove it:
   (a) $\displaystyle\sum_{k=1}^{\infty} f_k(x)$ converges pointwise on E, but not absolutely pointwise on E.
   (b) $\displaystyle\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on E, but not absolutely pointwise on E.
   (c) $\displaystyle\sum_{k=1}^{\infty} f_k(x)$ converges absolutely pointwise on E, but not uniformly on E.
   (d) $\displaystyle\sum_{k=1}^{\infty} f_k(x)$ converges absolutely uniformly on E, but the Weierstrass M-test fails.
6. Let $f_n(x) = (1 + \frac{x}{n})^n$ on $[0,R]$, for $R > 0$. Prove that f_n converges uniformly to e^x on $[0,R]$.
7. Find $f_n \in \mathcal{C}[0,1]$ with $||f_n||_\infty = 1$ such that no subsequence of $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on $[0,1]$.
8. Let $f_n(x) = \dfrac{nx}{1+nx}$ on $[0,1]$.
   (a) Find the pointwise limit, $f(x) = \lim_{n\to\infty} f_n(x)$.
   (b) Check if $\displaystyle\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \int_0^1 \lim_{n\to\infty} f_n(x)\, dx$.
   (c) Does f_n converge uniformly to f on $[0,1]$?
9. Let $f_n \in \mathcal{C}(E)$ for some $E \subset \mathbb{R}$ such that f_n converges to f uniformly on E. Prove that

$$f_n(x_n) \to f(x) \quad \text{for} \quad x_n \to x \in E\,.$$