1. (8 points) Answer each of the following questions completely. Use quantifiers such as \( \exists \) (there exists) and \( \forall \) (for all), when appropriate.

(a) A group action \( \phi \) of a group \( G \) on a set \( S \) is . . .

(b) If \( G \) acts on \( S \), then the orbit of the element \( s \in S \) is the set:

\[
\text{Orb}(s) = \{ \}
\]

In particular, it is a subset of (the group \( G \)) (the set \( S \)). [← circle one of these]

(c) If \( G \) acts on \( S \), then the stabilizer of an element \( s \in S \) is the set:

\[
\text{Stab}(s) = \{ \}
\]

In particular, it is a subset of (the group \( G \)) (the set \( S \)).

(d) If \( G \) acts on \( S \), then the fixed points of the action is the set:

\[
\text{Fix}(\phi) = \{ \}
\]

In particular, it is a subset of (the group \( G \)) (the set \( S \)).

(e) If \( G \) acts on \( S \), then the orbit-stabilizer theorem says that . . .
2. (8 points) Short answer:

(a) State the Fundamental Homomorphism Theorem.

(b) A subgroup $H$ of $G$ is normal iff its normalizer $N_G(H)$ is ________________.

(c) How many conjugacy classes does $S_5$ have? Write down exactly one element from each class in cycle notation.

(d) Make a list, as long as possible, of abelian groups of order $108 = 2^2 \cdot 3^3 = 108$, up to isomorphism. That is, every abelian group of order 108 should be isomorphic to exactly one group on your list.
3. (14 points) Give an example of each of the following.

(a) A group $G$ with isomorphic normal subgroups $H$ and $K$ such that $G/H$ and $G/K$ are non-isomorphic.

(b) A group $G$ with normal subgroup $N$ such that the direct product of $G/N$ and $N$ is not isomorphic to $G$.

(c) A subgroup $H \leq G$ whose normalizer is $N_G(H) = H$.

(d) A chain of subgroups $K \triangleleft H \triangleleft G$ such that $K$ is not normal in $G$.

(e) An automorphism (=isomorphism from a group to itself) of a group $G$ that is not the identity map.

(f) A group $G$ whose center $Z(G) := \{z \in G \mid zg = gz, \ \forall g \in G\}$ satisfies $\{e\} \leq Z(G) \leq G$.

(g) A nonabelian group such that all of its subgroups are normal.
4. (6 points) Let $G = C_5$ and $H = C_{24}$.

(a) How many homomorphisms are there from $G$ to $H$? Fully justify your answer.

(b) How many homomorphisms are there from $H$ to $G$? Fully justify your answer.

(c) Draw the subgroup lattice (or “Hasse diagram”) of $G = C_{12}$. 
5. (8 points) Let $S$ be the set of $2^3 = 8$ “binary triangles:”

$$S = \left\{ \begin{array}{c}
\triangle \begin{array}{c}
a \\
c \\
b
\end{array} : \ a, b, c \in \{0, 1\} \end{array} \right\}.$$

The group $G = D_3 = \{e, r, r^2, f, rf, r^2 f\}$ acts on $S$ via a homomorphism $\phi: D_3 \to \text{Perm}(S)$ where:

$\phi(r) =$ the permutation that rotates each triangle $120^\circ$ clockwise

$\phi(f) =$ the permutation that reflects each triangle about its vertical axis

(a) Draw the action diagram of this group action. What are the orbits of this action?

(b) What is $\text{Ker}(\phi)$? (That is, which specific subgroup of $D_3$ is it? Do not just give the definition of kernel.)
(c) For each of the following elements $s \in S$, find its stabilizer, $\text{Stab}(s)$.

- \[
\begin{array}{ccc}
0 & 0 & 0
\end{array}
\]

- \[
\begin{array}{ccc}
1 & 0 & 0
\end{array}
\]

- \[
\begin{array}{ccc}
0 & 0 & 1
\end{array}
\]

- \[
\begin{array}{ccc}
0 & 1 & 0
\end{array}
\]
6. (6 points) Prove that $A \times B \cong B \times A$. [Hint: Start by defining a map from $A \times B$ to $B \times A$.]