MthS 4120/6120: Abstract Algebra (Fall 2013)
Midterm 2

NAME:

Instructions: Answer each of the following questions completely. If something is unclear, or if you have any questions, then please ask. Good luck!

1. (8 points) Answer each of the following questions completely. Use quantifiers such as $\exists$ (there exists) and $\forall$ (for all), when appropriate.

   (a) A *group action* ϕ of a group G on a set S is ...

   (b) If G acts on S, then the *orbit* of the element $s \in S$ is the set:

   $$\mathrm{Orb}(s) = \Big\{ \qquad \Big\}.$$

   In particular, it is a subset of (the group G) (the set S). [$\longleftarrow$ circle one of these]

   (c) If G acts on S, then the *stabilizer* of an element $s \in S$ is the set:

   $$\mathrm{Stab}(s) = \Big\{ \qquad \Big\}.$$

   In particular, it is a subset of (the group G) (the set S).

   (d) If G acts on S, then the *fixed points* of the action is the set:

   $$\mathrm{Fix}(\phi) = \Big\{ \qquad \Big\}.$$

   In particular, it is a subset of (the group G) (the set S).

   (e) If G acts on S, then the *orbit-stabilizer theorem* says that ...

2. (8 points) Short answer:

   (a) State the Fundamental Homomorphism Theorem.

   (b) A subgroup H of G is normal iff its normalizer $N_G(H)$ is __________________________.

   (c) How many conjugacy classes does S_5 have? Write down exactly one element from each class in cycle notation.

   (d) Make a list, as long as possible, of abelian groups of order $108 = 2^2 \cdot 3^3 = 108$, up to isomorphism. That is, every abelian group of order 108 should be isomorphic to exactly one group on your list.

3. (14 points) Give an example of each of the following.

   (a) A group G with isomorphic normal subgroups H and K such that G/H and G/K are non-isomorphic.

   (b) A group G with normal subgroup N such that the direct product of G/N and N is not isomorphic to G.

   (c) A subgroup $H \leq G$ whose normalizer is $N_G(H) = H$.

   (d) A chain of subgroups $K \lhd H \lhd G$ such that K is not normal in G.

   (e) An automorphism (=isomorphism from a group to itself) of a group G that is not the identity map.

   (f) A group G whose center $Z(G) := \{z \in G \mid zg = gz,\ \forall g \in G\}$ satisfies $\{e\} \lneq Z(G) \lneq G$.

   (g) A nonabelian group such that all of its subgroups are normal.

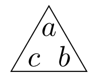
4. (6 points) Let $G = C_5$ and $H = C_{24}$.

   (a) How many homomorhpisms are there from G to H? Fully justify your answer.

   (b) How many homomorhpisms are there from H to G? Fully justify your answer.

   (c) Draw the subgroup lattice (or "Hasse diagram") of $G = C_{12}$.

5. (8 points) Let S be the set of $2^3 = 8$ "binary triangles:"

$S = \Bigg\{$ 

$: \quad a, b, c \in \{0, 1\} \Bigg\}$.

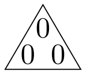
The group $G = D_3 = \{e, r, r^2, f, rf, r^2f\}$ acts on S via a homomorphism $\phi \colon D_3 \to \mathrm{Perm}(S)$ where:

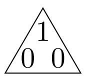
$$\phi(r) = \text{the permutation that rotates each triangle } 120^\circ \text{ clockwise}$$
$$\phi(f) = \text{the permutation that reflects each triangle about its vertical axis}$$

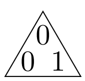
(a) Draw the *action diagram* of this group action. What are the orbits of this action?

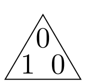
(b) What is $\mathrm{Ker}(\phi)$? (That is, which specific subgroup of D_3 is it? Do not just give the definition of kernel.)

(c) For each of the following elements $s \in S$, find its *stabilizer*, $\mathrm{Stab}(s)$.

- 

- 

- 

- 

6. (6 points) Prove that $A \times B \cong B \times A$. [*Hint*: Start by defining a map from $A \times B$ to $B \times A$.]