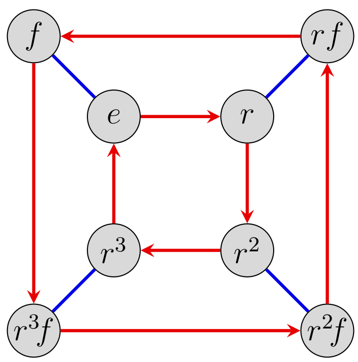
Read the following, which can all be found either in the textbook or on the course website.

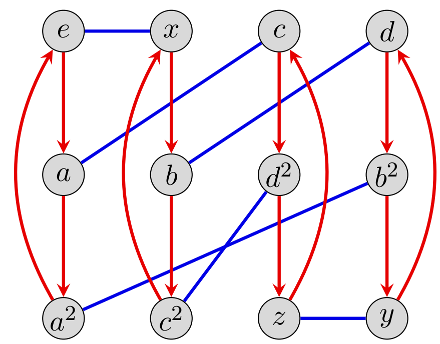
- Chapters 3 & 4 of *Visual Group Theory* (VGT).
- VGT Exercises 3.5–3.10, 3.12. 4.3–4.5, 4.10–4.14.

Write up solutions to the following exercises.

1. Draw a Cayley diagram for each of the 7 frieze groups using a minimal generating set. Make it clear what your generators are and write out a group presentation for each.

2. Shown below are the Cayley graphs of two groups: D_4 is on the left, and the right is called an "alternating group," denoted A_4.





   (a) Create a multiplication table for each group. For consistency, order the elements in D_4 by $(e, r, r^2, r^3, f, rf, r^2f, r^3f)$ and by $(e, x, y, z, a, a^2, b, b^2, c, c^2, d, d^2)$ in A_4.

   (b) Find the inverse of each element of each group.

   (c) Write out a group presentation for each group using the generators shown in the Cayley graph.

3. In each of the following multiplication tables, let e denote the identity element. Complete each table so its depicts a group. There may be more than one way to complete a table, in which case you need to give all possibilities.

	e	a
e		
a		

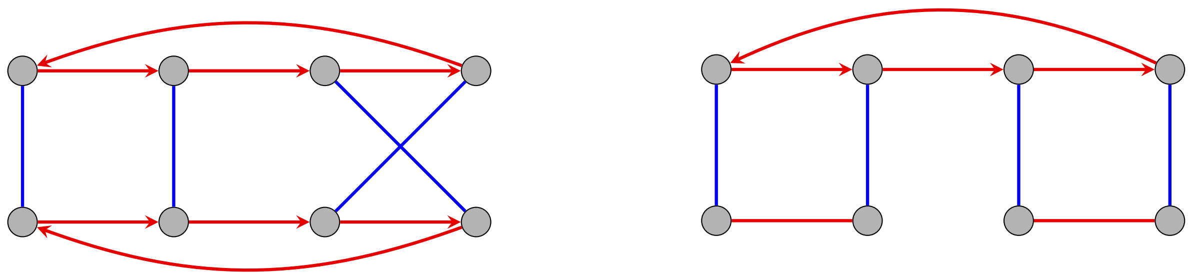
	e	a	b
e			
a			
b			

	e	a	b	c
e				
a		e		
b				
c				

4. Prove that an element cannot appear twice in the same column of a multiplication table.

5. Prove that every group has a unique identity action e, satisfying $ge = g = eg$ for every action g in G. [*Hint*: You need to prove both existence and uniqueness. For the latter, assume that e and f are both identity actions. Can you prove that $e = f$?]

6. In this exercise, you will see one reason why our unofficial definition of a group doesn't quite match our formal definition. Consider the two diagrams below, neither of which is a valid Cayley diagram.



   (a) Assuming that the arrows represent actions in a group, are the four rules in our unofficial definition of a group satisfied? For each rule, give a brief explanation why or why not.

   (b) Try to convert each of these diagrams into a multiplication table. What problem arises in each case?

   (c) Try to write a group presentation describing each "group." What problem arises in each case?

   (d) Explain what is wrong with these diagrams that caused the problems you encountered in Parts (b) and (c)?

   (e) Create another diagram that is not a valid Cayley diagram for the same reason.

7. Let $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the set of integers, rational numbers, and real numbers, respectively. Let $\mathbb{Z}^+$, $\mathbb{Q}^+$, and $\mathbb{R}^+$ denote the positive integers, rationals, and reals. Let $\mathbb{Z}^*$, $\mathbb{Q}^*$, and $\mathbb{R}^*$ denote the nonzero integers, rationals, and reals.

   (a) Which the above sets are groups under addition? For each one that is a group (excluding $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^*$), give a minimal generating set. For each one that is not, give an explicit reason for why it fails.

   (b) Which of the above sets are groups under multiplication? For each one that is a group (excluding $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^*$), give a minimal generating set if there is one. For each one that is not, give an explicit reason for why it fails.

   (c) Let $n\mathbb{Z}$ denote the set of all integers that are multiples of n. For what n is the set $n\mathbb{Z}$ a group under addition? Give a minimal generating set for each one that is a group.