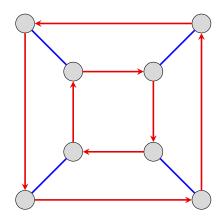
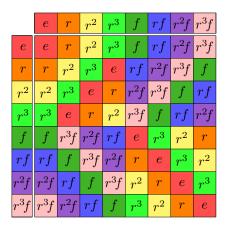
Read the following, which can all be found either in the textbook or on the course website.

- Chapters 6 of Visual Group Theory (VGT).
- VGT Exercises 6.6–6.9, 6.12, 6.17–6.20, 6.28–6.30.

Write up solutions to the following exercises.

1. A Cayley diagram and multiplication tables for the dihedral group D_4 are shown below.





Recall that Section 5.4.4 of VGT describes two algorithms for expressing a group G of order n as a set of permutations in S_n . One algorithm uses the Cayley diagram and the other uses the multiplication table. In this problem, you will explore this a bit further.

- (a) Label the vertices of the Cayley diagram from the set $\{1, ..., 8\}$ and use this to construct a permutation group isomorphic to D_4 , and sitting inside S_8 .
- (b) Label the entries of the multiplication table from the set $\{1, ..., 8\}$ and use this to construct a permutation group isomorphic to D_4 , and sitting inside S_8 .
- (c) Are the two groups you got in Parts (a) and (b) the same? (The answer will depend on your choice of labeling.) If "yes", then repeat Part (a) with a different labeling to yield a different group. If "no", then repeat Part (a) with a different labeling to yield the group you got in Part (b).
- 2. Find all subgroups of the following groups, and arrange them in a Hasse diagram, or subgroup lattice. Moreover, label each edge between $K \leq H$ with the index, [H:K].
 - (a) $C_{23} = \langle r \mid r^{23} = 1 \rangle;$
 - (b) $C_{24} = \langle r \mid r^{24} = 1 \rangle;$
 - (c) $\mathbb{Z}_3 \times \mathbb{Z}_3 = \{(a,b) \mid a,b \in \{0,1,2\}\};$
 - (d) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(a, b, c) \mid a, b, c \in \{0, 1\}\};$ (notationally easier to write elements as binary strings, e.g., abc instead of (a, b, c));
 - (e) $S_3 = \{e, (1\ 2), (2\ 3), (1\ 3), (1\ 2\ 3), (1\ 3\ 2)\};$
 - (f) $Q_4 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle$.

- 3. For each subgroup H of S_4 described below, write out all of its elements and determine what well-known group it is isomorphic to.
 - (a) $H = \langle (1\ 2), (3\ 4) \rangle;$
 - (b) $H = \langle (1\ 2)\ (3\ 4)\ ,\ (1\ 3)\ (2\ 4)\rangle;$
 - (c) $H = \langle (1\ 2), (2\ 3) \rangle;$
 - (d) $H = \langle (1\ 2), (1\ 3\ 2\ 4) \rangle;$
 - (e) $H = \langle (1\ 2\ 3), (2\ 3\ 4) \rangle$.
- 4. Prove the following, algebraically (that is, do not refer to Cayley diagrams):
 - (a) If \mathcal{H} is a collection of subgroups of G, then the intersection $\bigcap_{H \in \mathcal{H}} H$ is also a subgroup of G.
 - (b) For any subset $S \subseteq G$, the subgroup generated by S is defined as

$$\langle S \rangle := \{ s_1^{e_1} s_2^{e_2} \cdots s_k^{e_k} \mid s_i \in S, \ e_i \in \{-1, 1\} \}.$$

That is, $\langle S \rangle$ consists of all finite "words" that can be written using the elements in S and their inverses. Note that the s_i 's need not be distinct. Prove that

$$\langle S \rangle = \bigcap_{S \subseteq H \le G} H \,,$$

where the intersection is taken over all subgroups of G that contain S. [Hint: One way to prove that A = B is to show that $A \subseteq B$ and $B \subseteq A$.]

- 5. For a subgroup $H \leq G$ and element $x \in G$, the set $xH := \{xh \mid h \in H\}$ is a *left coset* of H.
 - (a) Prove that if $x \in H$, then xH = H. What is the interpretation of this statement in terms of the Cayley diagram?
 - (b) Prove that if $b \in aH$, then aH = bH.
 - (c) Prove that all left cosets have the same size. One way to do this is to prove that for any $x \in G$, the map

$$\varphi \colon H \longrightarrow xH$$
, $\varphi \colon h \longmapsto xh$

is a bijection.

- (d) Conclude that G is partitioned by the left cosets of H, all of which are equal size.
- 6. A subgroup H of G is normal if xH = Hx for all $x \in G$. Prove that if [G : H] = 2, then H is a normal subgroup of G. [Hint: Use the results of the previous problem.]