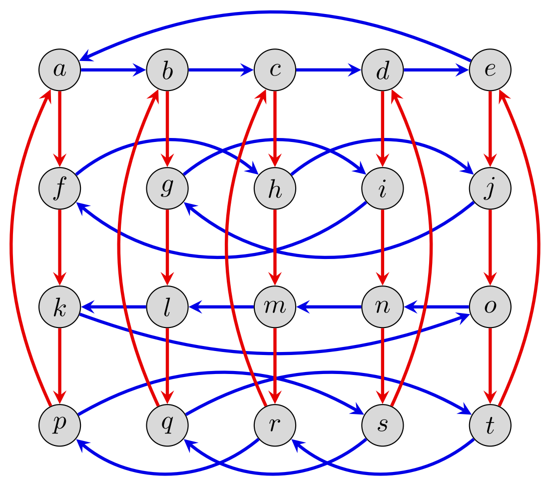
Read the following, which can all be found either in the textbook or on the course website.

- Chapters 7 of *Visual Group Theory* (VGT).
- VGT Exercises 7.3, 7.7–7.10, 7.12, 7.13, 7.17–7.20, 7.24–7.27, 7.30, 7.32–7.35.

Write up solutions to the following exercises.

1. Let G be the group whose Cayley diagram is shown below, and suppose e is the identity element. Consider the subgroups $H = \langle a \rangle = \{a, b, c, d, e\}$ and $K = \langle j \rangle = \{e, j, o, t\}$.



   Carry out the following steps for both of these subgroups.

   (a) Write G as a disjoint union of the subgroup's left cosets.

   (b) Write G as a disjoint union of the subgroup's right cosets.

   (c) Compute the normalizer of the subgroup.

   (d) If the subgroup (call it N) is indeed normal, then compute the quotient G/N, draw its Cayley diagram, and label the nodes appropriately.

2. All of the following statements are *false*. For each one, exhibit an explicit counter-example, and justify your reasoning. Assume that each $H_i \lhd G_i$ for $i = 1, 2$.

   (a) If every proper subgroup H of a group G is cyclic, then G is cyclic.

   (b) If $K \lhd H \lhd G$, then $K \lhd G$.

   (c) If $G_1 \cong G_2$ and $H_1 \cong H_2$, then $G_1/H_1 \cong G_2/H_2$.

   (d) If $G_1 \cong G_2$ and $G_1/H_1 \cong G_2/H_2$, then $H_1 \cong H_2$.

   (e) If $H_1 \cong H_2$ and $G_1/H_1 \cong G_2/H_2$, then $G_1 \cong G_2$.

3. Prove the following "subgroup criterion", which can be very useful when trying to show that a subset is indeed a subgroup: A nonempty subset H of a group G is a subgroup if and only if $xy^{-1} \in H$ holds for all $x, y \in H$.

4. Let A be a subset of a group G. The *centralizer* of A, denoted $C_G(A)$, is the set of all elements that commute with everything in A:

$$C_G(A) = \{g \in G \mid gag^{-1}a^{-1},\ \forall a \in A\}\,.$$

   (a) Prove that $C_G(A)$ is a subgroup of G.

   (b) If A is additionally a subgroup of G, prove that $C_G(A) \lhd N_G(A)$.

5. Prove that $A \times \{e_B\}$ is a normal subgroup of $A \times B$, where e_B is the identity element of B. That is, show first that it is a subgroup, and then that it is normal.

6. Partition the following groups into conjugacy classes:

   (a) $\mathbb{Z}_4$;

   (b) Q_4;

   (c) D_5;

   (d) A_4;

   (e) S_4.