Read the following, which can all be found either in the textbook or on the course website.

- Chapters 7 of *Visual Group Theory* (VGT).

Write up solutions to the following exercises.

1. Let $G$ be the group whose Cayley diagram is shown below, and suppose $e$ is the identity element. Consider the subgroups $H = \langle a \rangle = \{a, b, c, d, e\}$ and $K = \langle j \rangle = \{e, j, o, t\}$.

![Cayley Diagram](image)

Carry out the following steps for both of these subgroups.

(a) Write $G$ as a disjoint union of the subgroup’s left cosets.
(b) Write $G$ as a disjoint union of the subgroup’s right cosets.
(c) Compute the normalizer of the subgroup.
(d) If the subgroup (call it $N$) is indeed normal, then compute the quotient $G/N$, draw its Cayley diagram, and label the nodes appropriately.

2. All of the following statements are false. For each one, exhibit an explicit counter-example, and justify your reasoning. Assume that each $H_i \triangleleft G_i$ for $i = 1, 2$.

(a) If every proper subgroup $H$ of a group $G$ is cyclic, then $G$ is cyclic.
(b) If $K \triangleleft H \triangleleft G$, then $K \triangleleft G$.
(c) If $G_1 \cong G_2$ and $H_1 \cong H_2$, then $G_1/H_1 \cong G_2/H_2$.
(d) If $G_1 \cong G_2$ and $G_1/H_1 \cong G_2/H_2$, then $H_1 \cong H_2$.
(e) If $H_1 \cong H_2$ and $G_1/H_1 \cong G_2/H_2$, then $G_1 \cong G_2$.

3. Prove the following “subgroup criterion”, which can be very useful when trying to show that a subset is indeed a subgroup: A nonempty subset $H$ of a group $G$ is a subgroup if and only if $xy^{-1} \in H$ holds for all $x, y \in H$. 

4. Let \( A \) be a subset of a group \( G \). The centralizer of \( A \), denoted \( C_G(A) \), is the set of all elements that commute with everything in \( A \):

\[
C_G(A) = \{ g \in G \mid gag^{-1}a^{-1}, \forall a \in A \}.
\]

(a) Prove that \( C_G(A) \) is a subgroup of \( G \).
(b) If \( A \) is additionally a subgroup of \( G \), prove that \( C_G(A) \triangleleft N_G(A) \).

5. Prove that \( A \times \{e_B\} \) is a normal subgroup of \( A \times B \), where \( e_B \) is the identity element of \( B \). That is, show first that it is a subgroup, and then that it is normal.

6. Partition the following groups into conjugacy classes:

(a) \( \mathbb{Z}_4 \);
(b) \( \mathbb{Q}_4 \);
(c) \( D_5 \);
(d) \( A_4 \);
(e) \( S_4 \).