

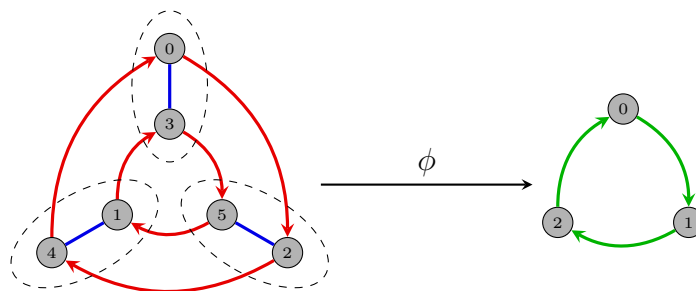
Read the following, which can all be found either in the textbook or on the course website.

- Chapters 8.1, 8.2 of *Visual Group Theory* (VGT).
- VGT Exercises 8.2, 8.5, 8.6, 8.9, 8.11, 8.13, 8.14.

Write up solutions to the following exercises.

1. Do the following steps for each mapping  $\phi: G \rightarrow H$  listed below.
  - (i) Determine whether  $\phi$  is a homomorphism.
  - (ii) Find  $\ker \phi := \{g \in G \mid \phi(g) = e\}$  and  $\text{im } \phi = \phi(G)$ .
  - (iii) Draw Cayley diagrams of the domain and codomain, and arrange them so one can “visually see” the cosets of  $\ker \phi$  in  $G$ . Draw dotted lines around these cosets.
  - (iv) Is the quotient  $G/\ker \phi$  a group? If so, what is it isomorphic to?

Here is an example of Step (iii) for the map  $\phi: \mathbb{Z}_6 \rightarrow \mathbb{Z}_3$ , defined by  $\phi(n) = n \pmod{3}$ .



Now, do steps (i)–(iv) for the following maps.

- (a) The map  $\phi: \mathbb{Z} \rightarrow \mathbb{Z}$  defined by  $\phi(n) = 4n$ .
  - (b) The map  $\phi: D_4 \rightarrow \mathbb{Z}_4$  defined by  $\phi(r^k f) = k$ .
  - (c) The map  $\phi: D_4 \rightarrow \mathbb{Z}_4$  defined by  $\phi(r^k f) = 2k$ .
  - (d) The map  $\phi: \mathbb{Z}_4 \rightarrow D_4$  defined by  $\phi(k) = r^k$ .
  - (e) The map  $\phi: D_4 \rightarrow V_4$  defined by  $\phi(r) = h$  and  $\phi(f) = v$ .
2. Prove that  $A \times B \cong B \times A$ .
  3. For each part below, list *all* homomorphisms with the given domain and codomain.
    - (a) Domain  $\mathbb{Z}_{15}$  and codomain  $\mathbb{Z}_4$ .
    - (b) Domain  $\mathbb{Z}_{412}$  and codomain  $\mathbb{Z}_{450}$ .
    - (c) Domain and codomain both  $\mathbb{Z}_4$ .
    - (d) Domain  $C_4$  and codomain  $V_4$ .
    - (e) Domain and codomain both  $V_4$ .

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4. Prove that there is no embedding  $\phi: \mathbb{Z}_n \hookrightarrow \mathbb{Z}$ .
5. Let  $H \leq G$ , and fix  $x \in G$ . Recall that we showed in class that  $xHx^{-1}$  is always a subgroup of  $G$ .
- (a) Prove additionally that  $xHx^{-1} \cong H$ . [*Hint*: Define a mapping from  $H$  to  $xHx^{-1}$  and prove that it is a homomorphism, one-to-one, and onto.]
  - (b) Use Part (a) to show that  $|xy| = |yx|$  for any  $x, y \in G$ .
6. Let  $\phi: G \rightarrow H$  be a homomorphism, and  $N \triangleleft H$ .
- (i) Show that the set  $\phi^{-1}(N) := \{g \in G \mid \phi(g) \in N\}$  is a subgroup of  $G$ .
  - (ii) Show that  $\phi^{-1}(N)$  is a *normal* subgroup of  $G$ .
  - (iii) Show by example that if  $M \triangleleft G$ , then  $\phi(M)$  need not be a normal subgroup of  $H$ .