Read the following, which can all be found either in the textbook or on the course website.

- Chapter 9.1 of *Visual Group Theory* (VGT).
- VGT Exercises 8.15–8.18, 9.17.

Write up solutions to the following exercises.

1. The commutator subgroup of a group $G$ is the subgroup

$$G' = \langle aba^{-1}b^{-1} \mid a, b \in G \rangle \, .$$

   (a) Prove that $G$ is abelian if and only if $G' = \{e\}$.

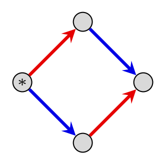
   (b) Prove that $G' \lhd G$. [*Hint*: Take a "commutator" $c = aba^{-1}b^{-1}$ and prove that $gcg^{-1} \in G'$.]

   (c) Prove that $G'$ is the intersection of all normal subgroups of $G$ that contain the set $C := \{aba^{-1}b^{-1} \mid a, b \in G\}$:

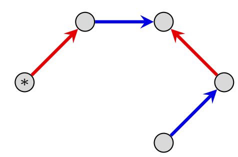
$$G' = \bigcap_{C \subseteq N \lhd G} N$$

   (d) If we quotient $G$ by $G'$, then we are in essence, "killing" all non-abelian parts of the Cayley diagram, as shown below:

$ab = ba$



$ab \neq ba$



   Prove algebraically that $G/G'$ is indeed abelian.

2. For each of the following groups $G$, compute its commutator subgroup $G'$ and its abelianization $G/G'$. Finally, draw the subgroup lattice of $G$ and circle every normal subgroup, and circle twice the one that is $G'$.

   (a) $V_4$

   (b) $D_3$

   (c) $Q_4$

3. Find the commutator subgroup of each of the following groups and compute its abelianization.

   (a) An abelian group $A$.

   (b) The alternating group $A_n$, for $n \geq 5$. [*Hint*: $A_n$ is a *simple group*, which means its only normal subgroups are $\langle e \rangle$ and $A_n$.]

   (c) The dihedral group $D_n$ for $n$ even.

   (d) The dihedral group $D_n$ for $n$ odd.

4. For each group $G$, find all automorphisms and make a multiplication table of $\mathrm{Aut}(G)$. What group is it isomorphic to?

    (a) $\mathbb{Z}_7$

    (b) $\mathbb{Z}_8$

    (c) $\mathbb{Z}_{10}$

    (d) $V_4$

    (e) $D_3$

5. Let $G$ act on a set $S$. Prove that $\mathrm{Stab}(s)$ is a subgroup of $G$ for every $s \in S$.

6. If $C_5$ acts on the set $S = \{A, B, C, D\}$, what will the action diagram be? Why?