1. For each of the following rings R, determine the zero divisors (right and left, if appropriate), and the set $U(R)$ of units.

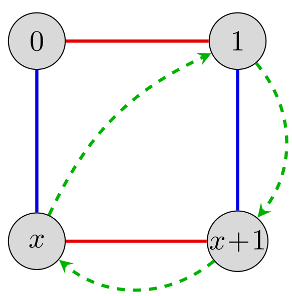
   (a) The set $\mathcal{C}^1$ of continuous real-valued functions $f \colon \mathbb{R} \to \mathbb{R}$.

   (b) The polynomial ring $\mathbb{R}[x]$.

   (c) $\mathbb{Z} \times \mathbb{Z}$, where addition and multiplication are defined componentwise.

   (d) $\mathbb{R} \times \mathbb{R}$, where addition and multiplication are defined componentwise.

2. The finite field $\mathbb{F}_4$ on 4 elements can be constructed as the quotient of the polynomial $\mathbb{Z}_2[x]$ by the ideal $I = (x^2 + x + 1)$ generated by the irreducible polynomial $x^2 + x + 1$. The figure below shows a Cayley diagram, and multiplication and addition tables for the finite field $\mathbb{Z}_2[x]/(x^2 + x + 1) \cong \mathbb{F}_4$.



$+$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\times$	1	x	$x+1$
1	1	x	$x+1$
x	x	$x+1$	1
$x+1$	$x+1$	1	x

   (a) Find a degree-3 polynomial $f \in \mathbb{Z}_2[x]$ that is irreducible over $\mathbb{Z}_2$, and a degree-2 polynomial $g \in \mathbb{Z}_3[x]$ that is irreducible over $\mathbb{Z}_3$. [*Hint*: Any polynomial with no roots in the prime field will work.]

   (b) Construct Cayley diagrams, addition, and multiplication tables for the finite fields

$$\mathbb{F}_8 \cong \mathbb{Z}_2[x]/(f) \qquad \text{and} \qquad \mathbb{F}_9 \cong \mathbb{Z}_3[x]/(g)\,.$$

3. For each of the following ideals, determine if it is prime and if it is maximal.

   (a) The ideal $I = (x)$ in the polynomial ring $R = \mathbb{Z}[x]$.

   (b) The ideal $I = (x)$ in the polynomial ring $R = \mathbb{R}[x]$.

   (c) The ideal $I = (x, y)$ in the multivariate polynomial ring $R = \mathbb{Z}[x, y]$.

   (d) The ideal $I = (x, y)$ in the multivariate polynomial ring $R = \mathbb{R}[x, y]$.

4. Prove that if a left ideal I of a ring R contains a unit, then $I = R$.

5. Prove the Fundamental Homomorphism Theorem (FHT) for rings: If $I \subseteq R$ is a two-sided ideal, then $R/I \cong \operatorname{im}\phi$. You may assume the FHT for groups.

6. Let R be a commutative ring with 1.

   (a) Prove that R is an integral domain if and only if 0 is a prime ideal.

   (b) Prove that an ideal $P \subseteq R$ is prime if and only if R/P is an integral domain.

   (c) Show that every maximal ideal is prime.