Chapter 3: Groups in science, art, and mathematics

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

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Overview

In the previous 2 chapters, we introduced groups and explored a few basic examples.

In this chapter, we shall discuss a few practical (but not complicated) applications.

We will see applications of group theory in 3 areas:

1. Science
2. Art
3. Mathematics

Our choice of examples is influenced by how well they illustrate the material rather than how useful they are.

Groups of symmetries

Intuitively, something is symmetrical when it looks the same from more than one point of view.

Can you think of an object that exhibits symmetry? Have we already seen some?

How does symmetry relate to groups? The examples of groups that we've seen so far deal with arrangements of similar things.

In Chapter 5, we will uncover the following fact (we'll be more precise later):

Cayley's Theorem

Every group can be viewed as a collection of ways to rearrange some set of things.

How to make a group out of symmetries

Groups relate to symmetry because an object's symmetries can be described using arrangements of the object's parts.

The following algorithm tells us how to construct a group that describes (or measures) a physical object's symmetry.

Algorithm 3.1

1. Identify all the parts of the object that are similar (e.g., the corners of an *n*-gon), and give each such part a different number.
2. Consider the actions that may rearrange the numbered parts, but leave the object it the same physical space. (This collection of actions forms a group.)
3. (Optional) If you want to visualize the group, explore and map it as we did in Chapter 2 with the rectangle puzzle, etc.

Comments

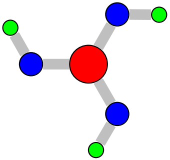
- We'll refer to the physical space that an object occupies as its footprint (this terminology does not appear in the text).
- Step 1 of Algorithm 3.1 numbers the object's parts so that we can track the manipulations permitted in Step 2. Each new state is a rearrangement of the object's similar parts and allows us to distinguish each of these rearrangements; otherwise we could not tell them apart.
- Not *every* rearrangement is valid. We are only allowed actions that maintain the object's physical integrity *and* preserve its footprint. For example, we can't rip two arms off a starfish and glue them back on in different places.
- Step 2 requires us to find *all* actions that preserve the object's footprint and physical integrity; not just the generators.
- However, if we choose to complete Step 3 (make a Cayley diagram), we must make a choice concerning generators. Different choices in generators may result in different Cayley diagrams.
- When selecting a set of generators, we would ideally like to select as small a set as possible. We can never choose too many generators, but we can choose too few. However, having "extra" generators only clutters our Cayley diagram.

Shapes of molecules

Because the shape of molecules impacts their behavior, chemists use group theory to classify their shapes. Let's look at an example.

The following figure depicts a molecule of Boric acid, $B(OH)_3$.

Note that a mirror reflection is *not* a symmetry of this molecule.

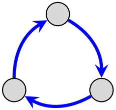


Exercise

Follow the steps of Algorithm 3.1 to find the group that describes the symmetry of the molecule and draw a possible Cayley diagram.

The group of symmetries of Boric acid has 3 actions requiring at least one generator. If we choose "120° clockwise rotation" as our generator, then the actions are:

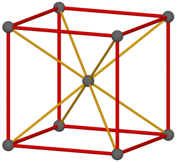
1. the *identity* (or "do nothing") action: e
2. 120° clockwise rotation: r
3. 240° clockwise rotation: r^2.

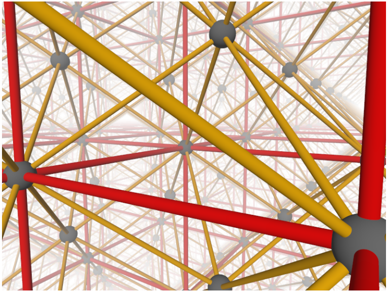


This is the cyclic group, C_3. (We'll discuss cyclic groups in Chapter 5.)

Crystallography

Solids whose atoms arrange themselves in a regular, repeating pattern are called crystals. The study of crystals is called crystallography.





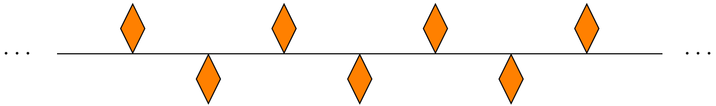
When chemists study such crystals they treat them as patterns that repeat without end. This allows a new manipulation that preserves the infinite footprint of the crystal and its physical integrity: translation.

In this case, the groups describing the symmetry of crystals are infinite. Why?

Frieze patterns

Crystals are patterns that repeat in 3 dimensions. Patterns that only repeat in 1 dimension are called frieze patterns. The groups that describe their symmetries are called frieze groups.

Frieze patterns (or at least finite sections of them) occur throughout art and architecture. Here is an example:



This frieze admits a new type of manipulation that preserves its footprint and physical integrity. This new action is called a glide reflection and consist of a horizontal translation (by the appropriate amount) followed by a vertical flip.

Note that for this pattern, a vertical flip all by itself does not preserve the footprint, and thus is not one of the actions of the group of symmetries.

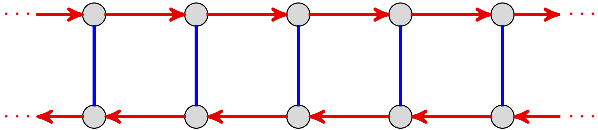
Exercise

Determine the group of symmetries of this frieze pattern and draw a possible Cayley diagram.

Frieze patterns

The group of symmetries of the frieze pattern on the previous slide turns out to be infinite, but we only needed two generators: horizontal flip and glide reflection.

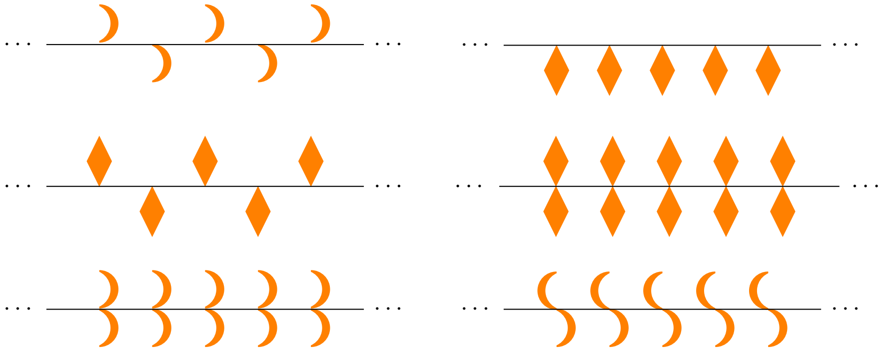
Here is a possible Cayley diagram:



Friezes, wallpapers, and crystals

- The symmetry of any frieze pattern can be described by one of 7 different infinite groups. Some frieze groups are *isomorphic* (have the same structure) even though the visual appearance of the patterns (and Cayley graphs) may differ.
- The symmetry of 2-dimensional repeating patterns, called "wallpaper patterns," has also been classified. There are 17 different wallpaper groups.
- There are 230 crystallographic groups, which describe the symmetries of 3-dimensional repeating patterns.

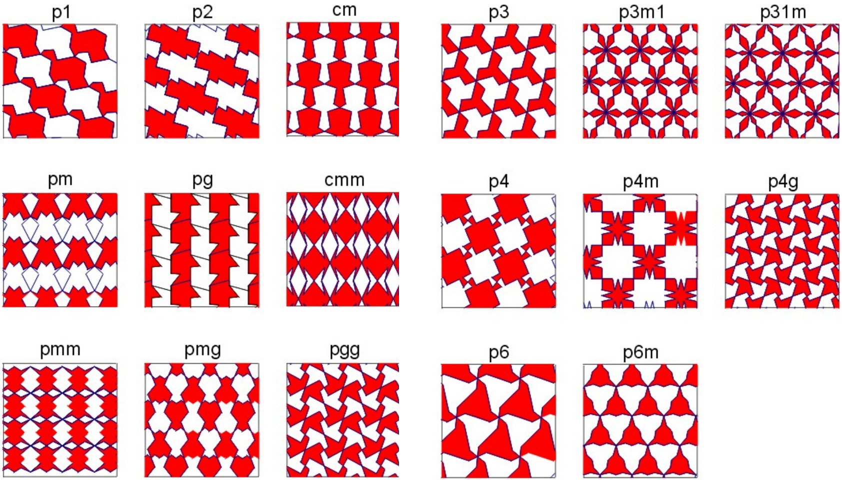
The 7 types of frieze patterns



Questions

- What basic types of symmetries (e.g., translation, reflection, rotation, glide reflection) do these frieze groups have?
- What are the (minimal) generators for the corresponding frieze groups?
- Which of these frieze patterns have isomorphic frieze groups?
- Which of these frieze groups are abelian?

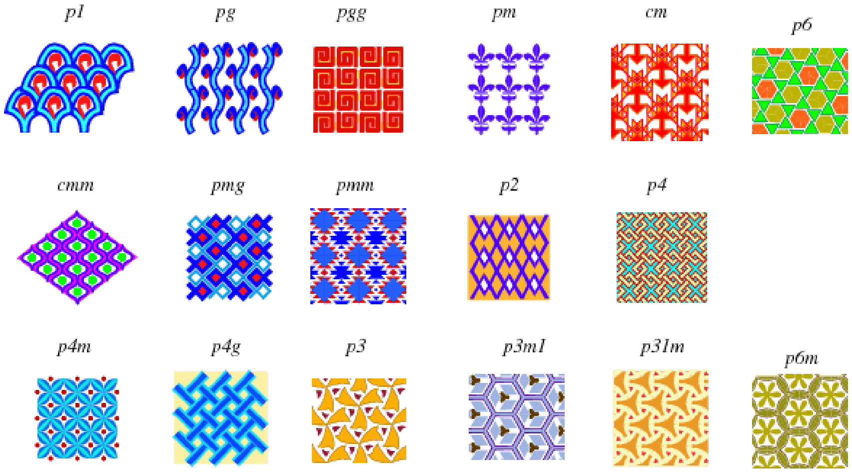
The 17 types of wallpaper patterns



Images courtesy of Patrick Morandi (New Mexico State University).

The 17 types of wallpaper patterns

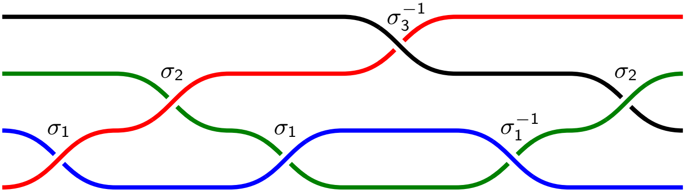
Here is another picture of all 17 wallpapers, with the official IUC notation for the symmetry group, adopted by the International Union of Crystallography in 1952.



Braid groups

Another area where groups arise in both art and mathematics is the study of braids.

This is best seen by an example. The following is a picture of an element (action) from the braid group $B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$:



The braid $b = \sigma_1\sigma_2\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2 = \sigma_1\sigma_2\sigma_3^{-1}\sigma_2$.

Do you see why the set of braids on n strings forms a group?

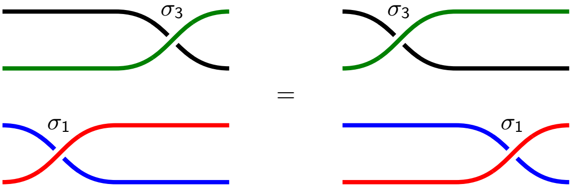
To combine two braids, just concatenate them.

Every braid is reversible – just "undo" each crossing. In the example above,

$$e = bb^{-1} = (\sigma_1\sigma_2\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2)(\sigma_2^{-1}\sigma_1\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}).$$

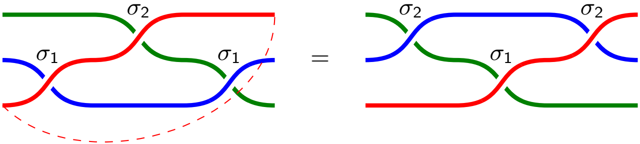
Braid groups

There are two fundamental relations in braid groups:



$$\sigma_i\sigma_j = \sigma_j\sigma_i$$

(if $|i-j| \geq 2$)



$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

We can describe the braid group B_4 by the following presentation:

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_3 = \sigma_3\sigma_1, \;\; \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \;\; \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \rangle.$$

We will study presentations more in the next chapter; this is just an introduction.