Math 2080: Differential Equations
Worksheet 7.6: Laplace's equation

NAME:

In this worksheet, you will solve the following three different instances of Laplace's equation on a square region, where $u(x, y)$ is defined for $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$:

(i) $\nabla^2 u = 0, \qquad u(x,0) = u(0,y) = u(\pi,y) = 0, \quad u(x,\pi) = \sin x.$

(ii) $\nabla^2 u = 0, \qquad u(x,0) = u(0,y) = u(x,\pi) = 0, \quad u(\pi,y) = \sin 2y.$

(iii) $\nabla^2 u = 0, \qquad u(x,0) = u(0,y) = 0, \quad u(x,\pi) = \sin x, \quad u(\pi,y) = \sin 2y.$

(a) Sketch the solutions to each of these three BVPs. *Hint: it is enough to sketch the boundaries, and then use the fact that the solutions are harmonic functions.*

(b) Solve the following BVP from (i) for $u(x, y)$:

$$\nabla^2 u = 0, \qquad u(x, 0) = u(0, y) = u(\pi, y) = 0, \quad u(x, \pi) = \sin x.$$

(c) Solve the following BVP from (ii) for $u(x, y)$:

$$\nabla^2 u = 0, \qquad u(x, 0) = u(0, y) = u(x, \pi) = 0, \quad u(\pi, y) = \sin 2y.$$

(d) Using superposition, solve following the BVP from (iii) for $u(x, y)$:

$$\nabla^2 u = 0, \qquad u(x, 0) = u(0, y) = 0, \quad u(x, \pi) = \sin x, \quad u(\pi, y) = \sin 2y.$$