Math 2080: Differential Equations
Worksheet 7.8: The two-dimensional wave equation

NAME:

1. Consider the following initial/boundary value problem for the 2D wave equation.

$$u_{tt} = c^2\nabla^2 u, \qquad u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0$$
$$u(x,y,0) = x(\pi - x)y(\pi - y), \quad u_t(x,y,0) = 1.$$

(a) Briefly describe a physical situation which this models. Be sure to explain the effect of the boundary conditions and the initial conditions. Sketch the initial displacement, $u(x,y,0)$.

(b) Assume that the solution has the form $u(x,y,t) = f(x,y)g(t)$. Find u_{xx}, u_{yy}, u_t, and u_{tt}.

(c) Plug $u = fg$ back into the PDE and separate variables (divide both sides by $c^2 fg$ to get the *eigenvalue problem*. Briefly justify why this quantity must be a constant, say λ. Write down an ODE for $g(t)$, and a PDE for $f(x, y)$ (the *Helmholz equation*). Include four boundary conditions for $f(x, y)$.

(d) You may assume that $\lambda = -(n^2 + m^2)$, and that the solution to the Helmholz equation is $f(x, y) = b_{nm} \sin nx \, \sin my$. Solve the ODE for $g(t)$.

(e) Find the general solution of the boundary value problem. It will be a superposition (infinite sum) of solutions $u_{nm}(x, y, t) = f_{nm}(x, y)g_{nm}(t)$.

(f) Find (or better yet, recall from your notes) the Fourier sine series for the following two functions:

$$p(x) = x(\pi - x), \quad \text{defined on } [0, \pi], \qquad\qquad r(x) = 1, \quad \text{defined on } [0, \pi].$$

(g) Find the particular solution to the initial value problem that additionally satisfies the initial conditions.

(h) What is the long-term behavior of $u(x, y, t)$, i.e., as $t \to \infty$. Give a mathematical *and* intuitive (physical) justification.