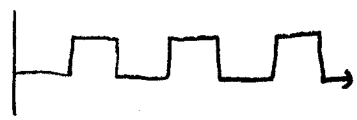
5. Laplace transforms

Laplace transforms are:

- Used to solve & analyze linear ODE's (control theory).
- Useful when the forcing term is discontinuous,

  e.g., step function  (Think: Force being turned on/off.)

Big idea:

$$
\begin{tikzcd}
\text{ODE} \arrow[r, "\mathcal{L}"] \arrow[d, dashed] & \text{Algebraic equation} \arrow[d] \\
\text{Sol'n to ODE} & \text{Sol'n to algebraic eq'n} \arrow[l, "\mathcal{L}^{-1}"']
\end{tikzcd}
$$

The Laplace transform is an operator: it inputs a function, and outputs a function.

Def: Suppose $f(t)$ is:
- defined for $0 < t < \infty$
- $|f(t)| \leq Ce^{at}$ for some C, a

Then the Laplace transform of f is the function $\mathcal{L}(f)$, where

$$\boxed{\mathcal{L}(f(t))(s) := F(s) = \int_0^\infty f(t)e^{-st}\,dt, \quad s > 0}$$

We often write $\mathcal{L}(f)$ as F, i.e., $f \xmapsto{\mathcal{L}} F$.

Example: Compute $\mathcal{L}(f)$, where $f(t) = e^{at}$.

$$F(s) = \int_0^\infty e^{at} e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \frac{e^{-(s-a)t}}{-(s-a)} \Big|_0^\infty$$

$$= \lim_{T\to\infty} \frac{e^{-(s-a)T}}{-(s-a)} + \frac{1}{s-a}$$

$$= \begin{cases} 0 & \text{if } s>a \\ \infty & \text{if } s\le a \end{cases} \quad \text{(i.e, the limit doesn't exist).}$$

Thus, $\boxed{\mathcal{L}(e^{at})(s) = \frac{1}{s-a}}$ if $s>a$ (and not defined otherwise).

Remark: Sometimes the domain of F is restricted.

e.g., $f(t)$ has domain $(-\infty, \infty)$

$F(s)$ has domain (a, ∞).

(Analogy: e^x has domain $(-\infty, \infty)$ but its inverse function $\ln x$ has domain $(0, \infty)$.)

Recall: Integration by parts (we'll need it!)

Let's rederive it: $(uv)' = u\,dv + du\,v$

$$u\,dv = (uv)' - v\,du$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Example: Let $f(t) = t$. Compute $\mathcal{L}(F)$.

$$F(s) = \int_0^\infty t\, e^{-st}\, dt. \qquad \text{Let } u = t \quad v = -\tfrac{1}{s} e^{-st}$$
$$du = dt \quad dv = e^{-st}\, dt$$

$$\int \underbrace{t}_{u}\, \underbrace{e^{-st}\, dt}_{dv} = \underbrace{-\tfrac{1}{s} t e^{-st}}_{uv} + \underbrace{\tfrac{1}{s} \int e^{-st}\, dt}_{-\int v\, du} = -\frac{t e^{-st}}{s} - \frac{e^{-st}}{s^2}.$$

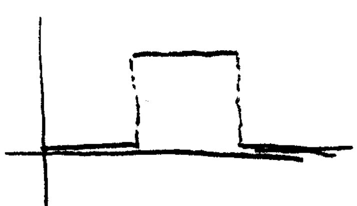
$$\text{So } F(s) = \lim_{T\to\infty} \left(-\frac{t e^{-st}}{s} - \frac{e^{-st}}{s^2} \right)\Bigg|_0^T = \lim_{T\to\infty} \left(\frac{-T e^{-sT}}{s} - \frac{e^{-sT}}{s^2} \right) - \left(0 - \frac{e^0}{s^2} \right) = \boxed{\frac{1}{s^2}}$$

Other common functions:

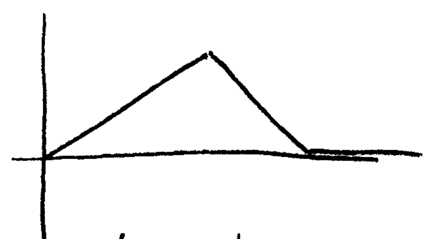
$$\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}, \quad \mathcal{L}(\sin bt)(s) = \frac{b}{s^2+b^2}, \quad \mathcal{L}(\cos bt)(s) = \frac{s}{s^2+b^2}.$$

We can also compute the Laplace transform of piecewise continuous & piecewise differentiable functions.

e.g.,

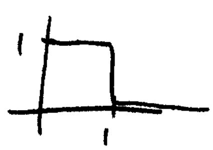


step function
(piecewise contin.)



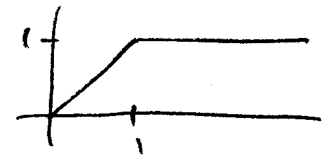
triangle wave
(piecewise diff'ble)

Example: Let $f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & t \geq 1 \end{cases}$



$$\mathcal{L}(f)(s) = F(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st}\, dt = -\tfrac{1}{s} e^{-st}\Big|_0^1 = \boxed{-\frac{e^{-s}}{s} + \frac{1}{s}}$$

Example: Let $f(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & 1 \le t < \infty \end{cases}$



We must break the interval into 2 parts:

$$\mathcal{L}(f(s)) = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 t\, e^{-st}\, dt + \int_1^\infty e^{-st}\, dt$$

Integrate by parts (exercise) $\longrightarrow$ $= \left(-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} - \frac{1}{s^2} \right) + \left(\frac{e^{-s}}{s} \right) = \boxed{\frac{1}{s^2} - \frac{e^{-s}}{s^2}}$.

Properties of the Laplace transform:

- $\mathcal{L}$ is linear
- $\mathcal{L}$ "turns derivatives into multiplication."

(i) Linearity: $\mathcal{L}(a f(t) + b\, g(t)) = a \mathcal{L}(f(t)) + b \mathcal{L}(g(t))$.

i.e., you can break apart sums & pull out constants.

($\mathcal{L}$ is a "linear operator.")

(ii) Turns derivatives into multiplication: $\boxed{\mathcal{L}(y'(t))(s) = sY(s) - y(0)}$

Proof: $\mathcal{L}(y')(s) = \int_0^\infty y'(t)\, e^{-st}\, dt$ [Let $u = e^{-st}$, $dv = y'(t)\, dt$]

$$= \underbrace{-e^{-st} y(t) \Big|_0^\infty}_{-y(0)} + \underbrace{s \int_0^\infty y(t)\, e^{-st}}_{sY(s)} = sY(s) - y(0). \checkmark$$

<u>Remark:</u> $\mathcal{L}(y'') = s\mathcal{L}(y') - y'(0)$ [by previous formula applied to y']

$$= s\big(sY(s) - y(0)\big) - y'(0)$$

$$= s^2 Y(s) - s\,y(0) - y'(0).$$

Formulas for $\mathcal{L}(y^{(k)})$ are derived similarly.

<u>More Laplace transform properties:</u>

(i) $\mathcal{L}(e^{at} f(t))(s) = F(s-a)$

(ii) $\mathcal{L}(t\,f(t))(s) = -F'(s)$.

(iii) $\mathcal{L}(t^n f(t))(s) = (-1)^n F^{(n)}(s)$ ← n^{th} derivative

<u>Example 1:</u> $f(t) = e^{2t}\cos 3t$ [Recall: $\mathcal{L}(\cos 3t) = \frac{s}{s^2+9}$]

Using (i), $F(s) = \mathcal{L}(e^{2t}\cos 3t) = \boxed{\frac{s-2}{(s-2)^2+9}}$

<u>Example 2:</u> $f(t) = t^2 e^{3t}$. Let $g(t) = e^{3t}$. [Recall $\mathcal{L}(t^2) = \frac{2}{s^3}$.]

Use (i): $\mathcal{L}(t^2 e^{3t}) = \boxed{\frac{2}{(s-3)^3}}$

<u>OR</u> Use (iii): $\mathcal{L}(t^2 e^{3t}) = (-1)^2 F''(s) = 1 \frac{d^2}{ds^2}\left(\frac{1}{s-3}\right) = \frac{2}{(s-3)^3}$.

<u>Using the Laplace transform to solve ODEs:</u>

<u>Example:</u> Consider the IVP: $y'' - y = e^{2t}$, $y(0) = 0$, $y'(0) = 1$.

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First step: Apply $\mathcal{L}$:

$$\begin{tikzcd} \boxed{\begin{array}{l} y''-y=e^{2t} \\ y(0)=0,\ y'(0)=1 \end{array}} \arrow[r, "\mathcal{L}"] \arrow[d, dashed] & s^2Y-1-Y=\frac{1}{s-2} \arrow[d, "\text{solve!}"] \\ \text{sol'n } y(t) & Y_{(s)}=\frac{1}{(s+1)(s-2)} \arrow[l, "\mathcal{L}^{-1}"'] \end{tikzcd}$$

$$\mathcal{L}(y'')-\mathcal{L}(y)=\mathcal{L}(e^{2t})$$

$$\to \left[s^2Y-s\,\cancel{y(0)}^{\,0}-y'(0)\right]-Y=\frac{1}{s-2}$$

$$s^2Y-1-Y=\frac{1}{s-2} \Rightarrow (s^2-1)Y=\frac{1}{s-2}+\frac{s-2}{s-2}=\frac{s-1}{s-2}$$

$$\Rightarrow \boxed{Y(s)=\frac{1}{(s+1)(s-2)}}$$

* The sol'n to the IVP is the (unique) function $y(t)$ whose Laplace transform is $Y(s)=\frac{1}{(s+1)(s-2)}$.

To find $y(t)$ [we say, to "invert" $Y(s)$], write

$$\frac{1}{(s+1)(s-2)}=\frac{A}{(s+1)}+\frac{B}{(s-2)}$$

Use partial fraction decomposition:

$$\frac{A}{(s+1)}\frac{(s-2)}{(s-2)}+\frac{B}{(s-2)}\frac{(s+1)}{(s+1)}=\frac{1}{(s+1)(s+2)} \Rightarrow \underbrace{(A+B)}_{=0}s+\underbrace{(B-2A)}_{=1.}=1$$

$$\Rightarrow \begin{cases} A+B=0 \\ B-2A=1 \end{cases} \Rightarrow A=-B \Rightarrow 3B=1 \Rightarrow \begin{array}{l} B=1/3 \\ A=-1/3. \end{array}$$

So, $$\frac{1}{(s+1)(s-2)}=\frac{-1/3}{s+1}+\frac{1/3}{s-2}$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{(s+1)(s-2)}\right) = \mathcal{L}^{-1}\left(\frac{-1/3}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{1/3}{s-2}\right)$$

$$= -\frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = \boxed{-\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}}$$

Problem: Partial fractions might not work:

Example: Compute $\mathcal{L}^{-1}\left(\frac{1}{s^2+4s+13}\right)$. [But $\frac{1}{s^2+4s+13}$ doesn't factor.]

Sol'n: Put it in the form $\frac{b}{(s-a)^2+b^2} = \mathcal{L}(e^{at}\sin bt)$.

"Complete the square"

$$\frac{1}{(s^2+4s+4)+9} = \frac{1}{(s+2)^2+3^2} = \frac{1}{3}\,\frac{3}{(s+2)^2+3^2} \xmapsto{\mathcal{L}^{-1}} \boxed{\frac{1}{3}e^{-2t}\sin 3t}$$

Example (when partial fractions fails)

Compute $\mathcal{L}^{-1}\left(\frac{1}{s^2+4s+13}\right)$. Problem: $s^2+4s+13$ doesn't factor.

Instead, put it in the form $\frac{1}{(s-a)^2+b^2}$, because

$$\mathcal{L}(e^{at}\sin bt) = \frac{b}{(s-a)^2+b^2}.$$

"Complete the square"

$$\frac{1}{s^2+4s+9} = \frac{1}{(s+2)^2+3^2} = \frac{1}{3}\,\frac{3}{(s+2)^2+3^2} \xmapsto{\mathcal{L}^{-1}} \frac{1}{3}e^{-2t}\sin 3t$$

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Comparison of old & new methods to solve:

$$y''-2y'-3y=0, \quad y(0)=1, \quad y'(0)=0.$$

Old method: Guess $y(t)=e^{rt} \ldots \quad e^{rt}(r^2-2r-3)=0$

$$\Rightarrow (r-3)(r+1)=0$$

$$\Rightarrow y(t)=C_1e^{3t}+C_2e^{-t}$$

Use IC's: $y(0)=C_1+C_2=1$

$y'(0)=3C_1-C_2=0 \quad \Rightarrow C_1=\frac{1}{4}, \quad C_2=\frac{3}{4}$

$$\Rightarrow y(t)=\frac{1}{4}e^{3t}+\frac{3}{4}e^{-t}$$

New method: $y''-2y'-3y=0, \quad y(0)=1, \quad y'(0)=0$

$$\mathcal{L}(y'')-2\mathcal{L}(y')-3\mathcal{L}(y)=0$$

$$[s^2Y-s\,y(0)-y'(0)]-2[sY-y(0)]-3Y=0$$

$$[s^2Y-s-0]-2[sY-1]-3Y=0 \Rightarrow (s^2-2s-3)Y=s-2$$

$$Y=\frac{s-2}{s^2-2s-3}=\frac{A}{s-3}\frac{(s+1)}{(s+1)}+\frac{B}{s+1}\frac{(s-3)}{(s-3)}=\frac{(A+B)s+(A-3B)}{(s+1)(s-3)}$$

$$\begin{cases}A+B=1\\A-3B=-2\end{cases} \Rightarrow \begin{matrix}A=1/4\\B=3/4\end{matrix} \Rightarrow Y(s)=\frac{1/4}{s-3}+\frac{3/4}{s+1}$$

$$y(t)=\frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)+\frac{3}{4}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)=\frac{1}{4}e^{3t}+\frac{3}{4}e^{-t}$$

<u>Summary / analysis:</u> Consider $ay''+by'+cy=f(t)$, $y(0)=x_0$, $y'(0)=v_0$

$$\mathcal{L}(ay''+by'+cy) = a\,\mathcal{L}(y'')+b\,\mathcal{L}(y')+c\,\mathcal{L}(y)$$
$$= a(s^2Y-s\,y(0)-y'(0))+b(sY-y(0))+cY$$
$$= (as^2+bs+c)Y - x_0(as+b) - a v_0 = F(s)$$

Thus, $$Y(s) = \underbrace{\frac{F(s)}{as^2+bs+c}}_{\text{"state-free sol'n"}} + \underbrace{\frac{(as+b)x_0+av_0}{as^2+bs+c}}_{\text{"input-free sol'n"}}$$

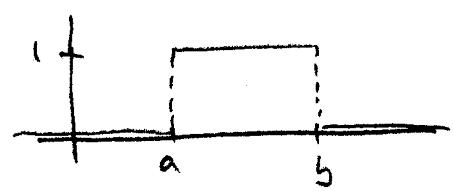
i.e., $Y(s) = Y_s(s) + Y_i(s)$, where

$Y_s(s)$ doesn't depend on the IC's, $y(0)=x_0$, $y'(0)=v_0$

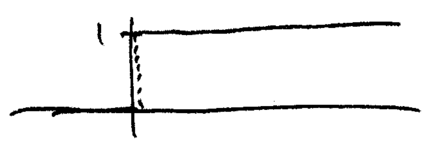
$Y_i(s)$ doesn't depend on the forcing term $f(t)$.

<u>Discontinuous forcing terms:</u>

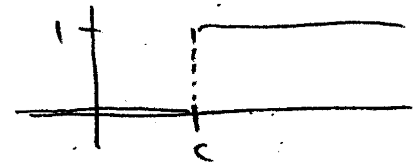
- Interval function: $H_{ab}(t) = \begin{cases} 0 & t<a \\ 1 & a\le t<b \\ 0 & b\le t<\infty \end{cases}$

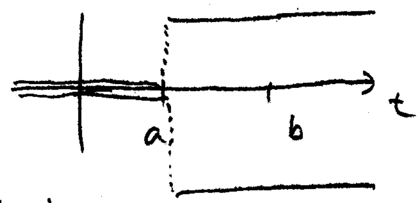


- Heavyside function: $H(t) = \begin{cases} 0 & t<0 \\ 1 & t\ge 0 \end{cases}$



- Shifted Heavyside function: $H_c(t) = \begin{cases} 0 & t<c \\ 1 & t\ge c \end{cases}$

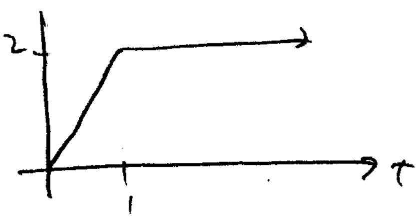




Remarks: • $H_c(t) = H(t-c)$

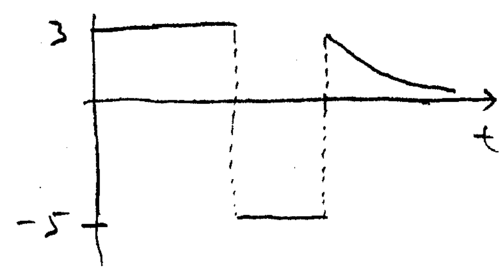
• $H_{ab}(t) = H_a(t) - H_b(t) = H(t-a) - H(t-b)$

* Piecewise continuous functions can be written using Heavyside functions.



Example 1: $f(t) = \begin{cases} 2t & 0 \le t < 1 \\ 2 & t \geq 1 \end{cases}$

$$f(t) = 2t\, H_{01}(t) + 2\, H_1(t)$$
$$= 2t\left[H(t) - H(t-1)\right] + 2\, H(t-1) = \boxed{2t\, H(t) - 2(t-1)\, H(t-1)}$$



Example 2: $f(t) = \begin{cases} 3 & 0 \le t < 4 \\ -5 & 4 \le t < 6 \\ e^{7-t} & 6 \le t < \infty \end{cases}$

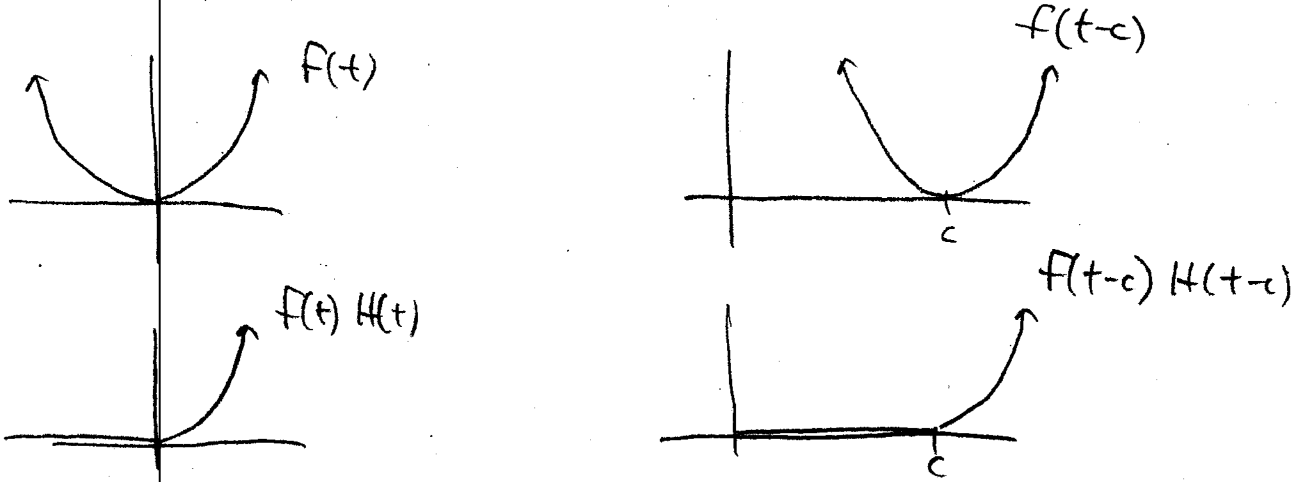
$$f(t) = 3\, H_{04}(t) - 5\, H_{46}(t) + e^{7-t} H_6(t)$$
$$= 3\left[H(t) - H(t-4)\right] - 5\left[H(t-4) - H(t-6)\right] + e^{7-t}\left[H(t-6)\right]$$
$$= 3\, H(t) - 8\, H(t-4) + 5\, H(t-6) + e^{7-t} H(t-6)$$

Motivation: Writing piecewise continuous functions in this form is convenient for taking their Laplace transforms.

Remark: $\mathcal{L}(H_{ab}(t)) = \int_a^b e^{-st}\,dt = \left.\dfrac{e^{-st}}{-s}\right|_a^b = \dfrac{e^{-as} - e^{-bs}}{s}$

Fact: $\boxed{\mathcal{L}\big(f(t-c)\,H(t-c)\big) = e^{-cs}\,F(s)}$

How to interpret this:



Remarks: • $\mathcal{L}(f(t)) = \mathcal{L}(f(t)\,H(t))$ always (why?)

• The above "Fact" is in a sense "dual" to the property that $\mathcal{L}(e^{ct} f(t)) = F(s-c)$:

"multiplication by an exponential in the t-domain corresponds to a shift in the s-domain, and vice-versa!"

<u>Example</u>: Compute the Laplace transform of $f(t) = \begin{cases} 2t & 0 \le t < 1 \\ 2 & t \ge 1 \end{cases}$

Recall: $f(t) = 2t\,H(t) - 2(t-1)\,H(t-1)$

$$\Rightarrow F(s) = 2\mathcal{L}(t\,H(t)) - 2\,\mathcal{L}((t-1)\,H(t-1)) = \boxed{\frac{2}{s^2} - \frac{2}{s^2}e^{-s}}$$

More practice using $\mathcal{L}(f(t-c)\,H(t-c)) = e^{-cs}F(s)$:

- $\mathcal{L}\{(t-3)^2 H(t-3)\} = e^{-3s}F(s)$

  $\hookrightarrow f(t-3) = (t-3)^2 \Rightarrow f(t) = f((t+3)-3) = ((t+3)-3)^2 = t^2$

  $\Rightarrow F(s) = \frac{1}{s}$

- $\mathcal{L}\{t^2 H(t-3)\} = e^{-3s}F(s)$

  $\hookrightarrow f(t-3) = t^2 \Rightarrow f(t) = f((t+3)-3) = (t+3)^2 = t^2 + 6t + 9$

  $\Rightarrow F(s) = \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}$

- $\mathcal{L}\{e^{t-1} H(t-1)\} = e^{-s}F(s)$

  $\hookrightarrow f(t-1) = e^{t-1} \Rightarrow f(t) = f((t+1)-1) = e^{(t+1)-1} = e^t$

  $\Rightarrow F(s) = \frac{1}{s-1}$

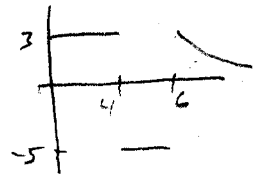
- $\mathcal{L}\{e^{7-t} H(t-6)\} = e^{-6s}F(s)$

  $\hookrightarrow f(t-6) = e^{7-t} \Rightarrow f(t) = f((t+6)-6) = e^{7-(t+6)} = e^{1-t} = e^1 e^{-t}$

  $\Rightarrow F(s) = e \cdot \frac{1}{s+1}$

<u>Trick</u>: Given $f(t-c)$ plug in $t+c$ for t to find $f(t)$.

<u>Example</u>: Find $F(s)$, where $f(t) = \begin{cases} 3 & 0 \le t < 4 \\ -5 & 4 \le t < 6 \\ e^{7-t} & 6 \le t < \infty \end{cases}$

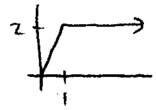


<u>Recall</u>: $f(t) = 3H(t) - 8H(t-4) + 5H(t-6) + e^{7-t} H(t-6)$

$$F(s) = \frac{3}{s} - \frac{8}{s}e^{-4s} + \frac{5}{s}e^{-6s} + \frac{1}{s+1}e^{1-6s} = \frac{3 - 8e^{-4s} + 5e^{-6s}}{s} + \frac{e^{1-6s}}{s+1}.$$

<u>Example</u>: Solve the IVP: $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 1$,

where $f(t) = \begin{cases} 2t & 0 \le t \le 1 \\ 2 & t > 1 \end{cases}$



<u>Recall</u>: $f(t) = 2t\, H(t) - 2(t-1)H(t-1)$

$$F(s) = \frac{2}{s^2} - \frac{2}{s^2}e^{-s}.$$

Take $\mathcal{L}$ of both sides of the ODE:

$$[s^2 Y - s y(0) - y'(0)] + Y = \frac{2 - 2e^{-s}}{s^2}$$

$$s^2 Y - 1 + Y = \frac{2 - 2e^{-s}}{s^2}$$

$$(s^2+1)Y = \frac{2-2e^{-s}}{s^2} + 1 \Rightarrow Y(s) = \frac{2 - 2e^{-s}}{s^2(s^2+1)} + \frac{1}{s^2+1}$$

Partial fractions: $\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$

$$Y(s) = \frac{2}{s^2} - \frac{2}{s^2+1} - \frac{2e^{-s}}{s^2} + \frac{2e^{-s}}{s^2+1} + \frac{1}{s^2+1}$$

$$= \frac{2}{s^2} - \frac{2e^{-s}}{s^2} - \frac{1}{s^2+1} + \frac{2e^{-s}}{s^2+1}$$

$$y(t) = 2\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - 2\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) + 2\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+1}\right)$$

$$= 2t - 2(t-1)H(t-1) - \sin t + 2\sin(t-1)H(t-1)$$

$$= [2t - \sin t] + [2\sin(t-1) - 2(t-1)]H(t-1)$$

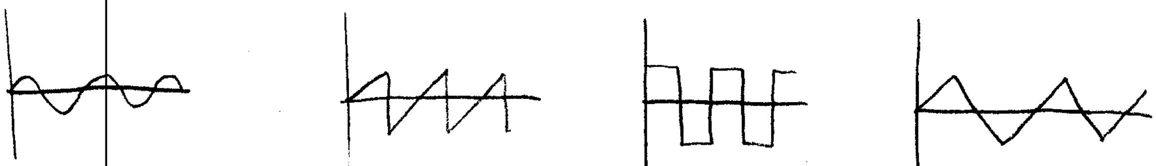
$$= \begin{cases} 2t - \sin t & 0 \le t < 1 \\ 2 + 2\sin(t-1) - \sin t & t \ge 1 \end{cases}$$

* This is the <u>unique</u> sol'n to the IVP $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 1$.

Periodic forcing terms:

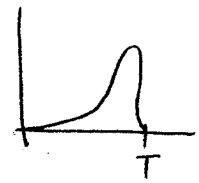
- Suppose $f(t)$ is periodic. We want to compute $F(s) = \mathcal{L}(f(t))(s)$.

e.g.,



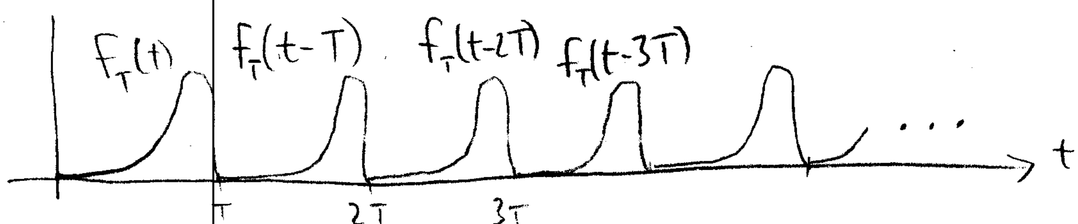
sine wave sawtooth wave square wave triangle wave.

<u>Approach</u>: Consider the "window" $f_T(t)$, defined on $0 \le t < T$, and extended to be periodic



Then, the function $f(t) = \begin{cases} f_T(t) & 0 \le t < T \\ f_T(t-kT) & kT \le t < (k+1)T \end{cases}$

is periodic:



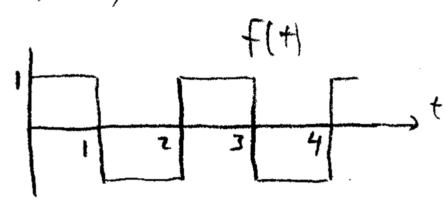
<u>Remark</u>: $f_T(t-kT) = f_T(t-kT)H(t-kT)$, so $f(t) = \sum_{k=0}^{\infty} f_T(t-kT)H(t-kT)$.

<u>key point</u>: If $|x|<1$, then $1 + x + x^2 + x^3 + \ldots = \boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$

Apply this to $F(s) = \sum_{k=0}^{\infty} \mathcal{L}\{f_T(t-kT)H(t-kT)\} = \sum_{k=0}^{\infty} e^{-kTs} F_T(s)$

$$= F_T(s) \sum_{k=0}^{\infty} e^{-kTs} = F_T(s) \sum_{k=0}^{\infty} (e^{-sT})^k = \boxed{F_T(s) \frac{1}{1-e^{-sT}}}$$

<u>Example</u>: Solve the IVP $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 0$, where $f(t)$ is the following square wave, of period $T = 2$



First, compute $F(s)$:

$$f_T(t) = H_{01}(t) - H_{12}(t) = [H(t) - H(t-1)] - [H(t-1) - H(t-2)]$$

$$= H(t) - 2H(t-1) + H(t-2)$$

$$F_T(s) = \mathcal{L}(H(t)) - 2\mathcal{L}(H(t-1)) + \mathcal{L}(H(t-2))$$

$$= \frac{1}{s} - \frac{2}{s}e^{-s} + \frac{1}{s}e^{-2s} = \boxed{\frac{(1-e^{-s})^2}{s}}$$

Now, $F(s) = F_T(s)\dfrac{1}{1-e^{-2s}} = \dfrac{F_T(s)}{(1-e^{-s})(1+e^{-s})} = \dfrac{(1-e^{-s})(1-e^{-s})}{s(1-e^{-s})(1+e^{-s})} = \boxed{\dfrac{(1-e^{-s})}{s(1+e^{-s})}}$

Back to the IVP: $\mathcal{L}(y'') + \mathcal{L}(y) = F(s)$

$$[s^2Y - sy(0) - y'(0)] + Y = \frac{1-e^{-s}}{s(1+e^{-s})}$$

$$(s^2+1)Y = \frac{1-e^{-s}}{s(1+e^{-s})} \Rightarrow \boxed{Y(s) = \frac{1-e^{-s}}{s(s^2+1)(1+e^{-s})}}$$

<u>Simplify this:</u> $Y(s) = \underbrace{\dfrac{1}{s(s^2+1)}}_{\frac{1}{s} - \frac{s}{s^2+1}} \cdot \underbrace{\dfrac{1-e^{-s}}{1+e^{-s}}}_{\text{need to further simplify.}}$

$$\frac{1-e^{-s}}{1+e^{-s}} = \frac{-(1+e^{-s})}{1+e^{-s}} + \frac{2}{1+e^{-s}} = 2\left(\frac{1}{1-(-e^{-s})}\right) = 2\sum_{n=0}^{\infty}(-1)^n e^{-ns}$$

Using $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$, we have $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \displaystyle\sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$

Thus, $Y(s) = \left(\dfrac{1}{s} - \dfrac{s}{s^2+1}\right)\left(-1 + 2\displaystyle\sum_{n=0}^{\infty}(-1)^n e^{-ns}\right)$

$$= \underbrace{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)}_{\text{Call this } G(s).}\left(+1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-ns}\right) = G(s) + 2\sum_{n=1}^{\infty}(-1)^n e^{-ns}G(s).$$

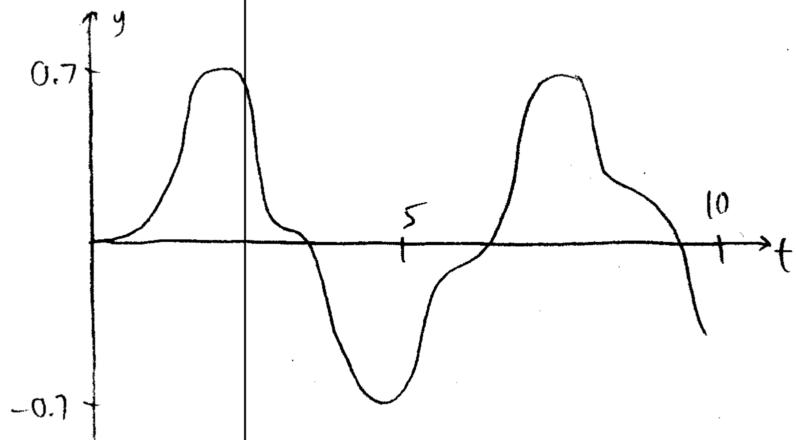
Note that $\boxed{g(t) = 1 - \cos t}$

Apply $\mathcal{L}\{g(t-n)H(t-n)\} = e^{-ns}G(s)$.

$$Y(s) = G(s) + 2\sum_{n=1}^{\infty}(-1)^n e^{-ns}G(s) \Longrightarrow y(t) = g(t) + 2\sum_{n=1}^{\infty}(-1)^n g(t-n)H(t-n)$$

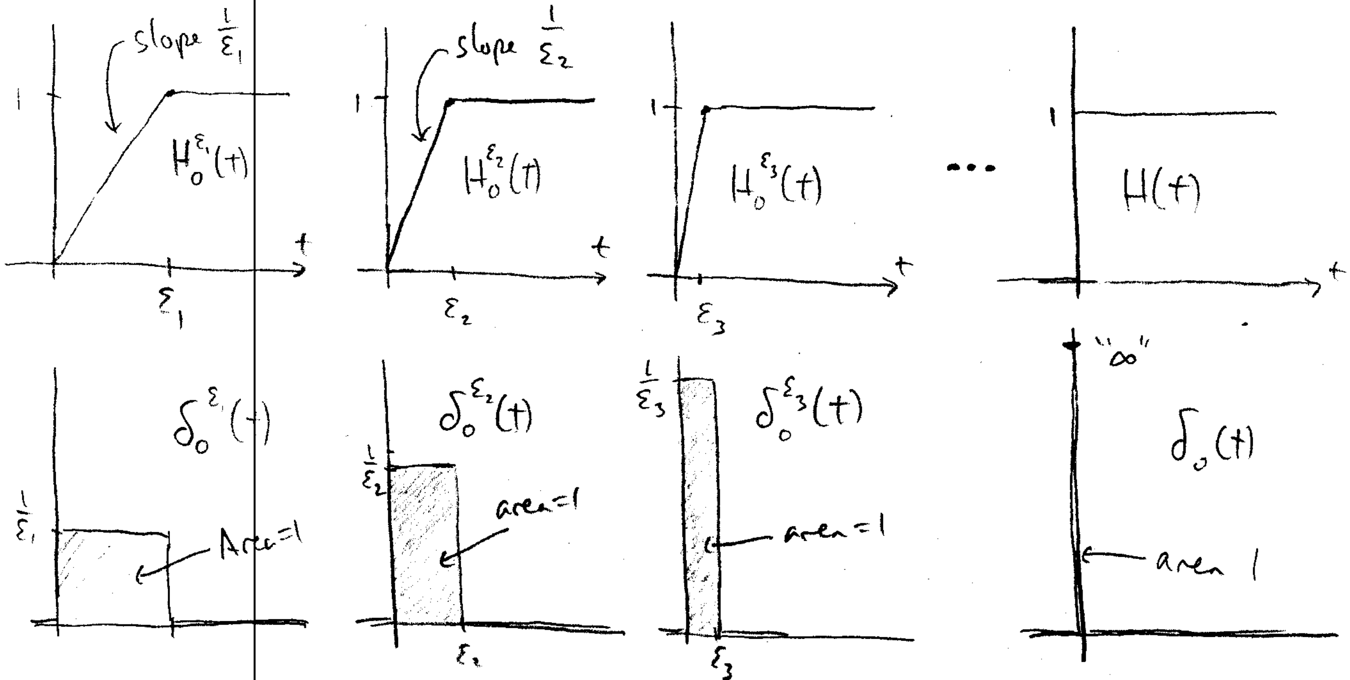
$$\boxed{y(t) = (1-\cos t)H(t) + 2\sum_{n=1}^{\infty}(-1)^n\left[1-\cos(t-n)\right]H(t-n)}$$

This is a superposition of infinitely many waves:



Question: What is the "derivative" of the Heavyside function?

Technically, it's not defined, but what "should" it be?



Def: The delta function is $\delta_p(t) = \lim_{\varepsilon\to 0} \delta_p^{\varepsilon}(t)$; $\quad \delta_p(t) = \begin{cases} 0 & t \neq p \\ \infty & t = p \end{cases}$

Technically, it's not really a function, but it's useful!

It has infinite height, infintesimal with, and integral 1.

Properties of the Delta function:

- $\int_{-\infty}^{\infty} \delta_p(t)\, dt = 1 \implies \int_{-\infty}^{\infty} \delta_p(t) f(t)\, dt = f(p)$
- $\mathcal{L}(\delta_0) = 1 \implies \mathcal{L}(\delta_p) = e^{-sp}$

So now we can take the inverse Laplace transform of a constant or an exponential.

* The Delta function models a unit impulse force (finite force over an infintesimal time interval (or realistically, a very small interval). e.g., Exerting a force by hitting something with a hammer.

Example: Solve the IVP $y'' + 2y' + 2y = \delta_0(t)$, $y(0) = 0$, $y'(0) = 0$.

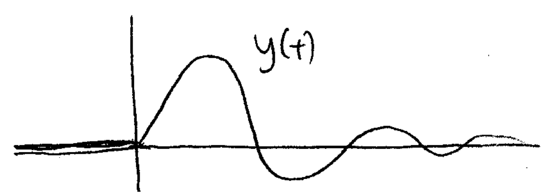
First, take the Laplace transform: $\mathcal{L}(y'' + 2y' + 2y) = \mathcal{L}(\delta_0(t))$

$$(s^2 + 2s + 2)Y = 1$$

$$\implies Y(s) = \frac{1}{s^2 + 2s + 2} = \frac{1}{(s+1)^2 + 1} \longrightarrow \boxed{y(t) = e^{-t} \sin t}$$

Remark: $y(0) = 0$ but $y'(0) = 1$. (This doesn't match the initial conditions!)

This is okay, because $y(t)$ is only defined for $t > 0$, so what we really have is



$$H(t)\, y(t) = \begin{cases} 0 & t \le 0 \\ e^{-t} \sin t & t > 0 \end{cases}$$

This isn't even differentiable at $t = 0$, so technically, the derivative isn't defined. But $\lim_{t \to 0^-} \frac{d}{dt}\left(H(t)\, y(t) \right) = 0$, and that's "good enough."

Convolution

Motivation: Laplace transforms are hard to compute.

We like to compute new ones from old

e.g., $\mathcal{L}(e^{ct} f(t)) = F(s-c)$, $\quad \mathcal{L}(f+g) = F(s) + G(s)$.

Question: Is there a formula for $\mathcal{L}(f(t)\, g(t))$?

No!

"Dual" question: Is there a formula for $\mathcal{L}^{-1}(F(s)\, G(s))$.

Yes! $\mathcal{L}^{-1}(F(s)\, G(s)) = f(t) * g(t) := \int_0^t f(u)\, g(t-u)\, du$

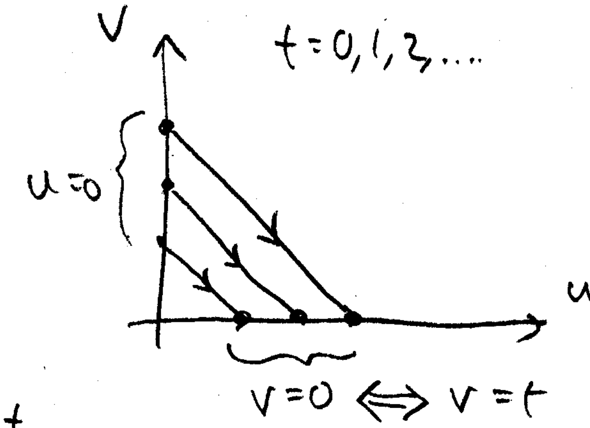
Proof: $F(s)\, G(s) = \left(\int_0^\infty e^{-su} f(u)\, du\right)\left(\int_0^\infty e^{-sv} g(v)\, dv\right)$

$$= \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u)\, g(v)\, du\, dv \qquad (*)$$

Change variables: let $\begin{cases} t = u+v \\ u = u \end{cases}$ $\quad du\, dv = \underbrace{\frac{\partial(u,v)}{\partial(u,t)}}\, du\, dt$

$$J = \begin{vmatrix} u_u & u_t \\ v_u & v_t \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

So $(*)$ becomes $\int_0^\infty \int_0^t e^{-st} f(u)\, g(t-u)\, du\, dv$



Change order of integration:

(*) becomes

$$\int_0^\infty e^{-st}\left(\int_0^t f(u)\, g(t-u)\, du\right) dt$$

$$= \mathcal{L}\left(f(t) * g(u)\right). \quad \checkmark$$

<u>Ex 1</u>: $t^2 * t = \int_0^t u^2(t-u)\, du = \left(\frac{u^3}{3}t - \frac{u^4}{4}\right)\Big|_0^t$

$$= \frac{t^4}{3} - \frac{t^4}{4} = \boxed{\frac{t^4}{12}}$$

$$\mathcal{L}(t^2 * t) = \mathcal{L}\left(\frac{t^4}{12}\right) = \frac{1}{12}\cdot\frac{24}{s^5} = \frac{2}{s^5} = \frac{2}{s^3}\cdot\frac{1}{s^2} = \mathcal{L}(t^2)\mathcal{L}(t). \checkmark$$

<u>Ex 2</u>: $f(t) * 1 = \int_0^t f(u)\cdot 1\, dt = \int_0^t f(u)\, du.$

<u>Properties</u>:
- $f * g = g * f$ (since $FG = GF$)
- $f * (g * h) = (f * g) * h$

Convolutions also arise in ODE outside of Laplace transforms.

<u>Example</u>: Suppose a company dumps radioactive (decaying) waste at a <u>rate</u> $f(t)$

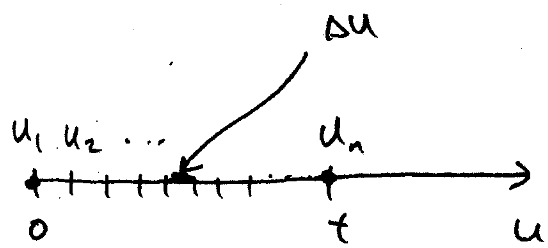
Amount dumped in $[t_i, t_{i+1}]$ is $\approx f(t)\cdot \Delta t$

<u>Problem</u>: Start dumping at $t=0$. At time t, how much waste remains?

That is, find $M(t)$ = mass of waste pile.

let M_0 = initial amt dumped

k = decay const.



Amt dumped in $[u_i, u_{i+1}] \approx f(u_i)\,\Delta u$

By time t, it's decayed to $f(u_i)\,\Delta u \cdot e^{-k(t-u_i)}$.

Total amt left at time $t = M(t) \approx \sum_{i=1}^{n} f(u_i)\, e^{-k(t-u_i)}\,\Delta u$

letting $\Delta u \to 0 \rightsquigarrow M(t) = \int_0^t f(u)\, e^{-k(t-u)}\, du$

$$= \boxed{f(t) * e^{-kt}}$$

how much was added

intrinsic decay rate of material.

<u>Modification</u>: Dump garbage (decay rate $k=0$):

$M(t) = f(t) * 1 = \int_0^t f(u)\, du$.

Situation where $k > 0$: Investment where $f(t)$ = rate of money added.

Situation where growth rate is linear (approx):

Let $M(t)$ = mass of chickens on a chicken farm.

Farmer wants to know how much feed to buy.

$M(t) \approx f(t) * t$

growth rate of colony

(birth rate) - (death rate)