Lecture 3.3: The method of undetermined coefficients

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Math 2080, Differential Equations

Recall our two "big ideas" for 1st order linear ODEs

Big idea 1

Suppose a homogeneous ODE $y' + a(t)y(t) = 0$ has solutions $y_1(t)$ and $y_2(t)$. Then

$$C_1 y_1(t) + C_2 y_2(t)$$

is a solution for any constants C_1 and C_2.

Big idea 2

Consider an inhomogeneous ODE $y' + a(t)y(t) = f(t)$. If $y_p(t)$ is *any* particular solution, and $y_h(t)$ is the general solution to the related "homogeneous equation", $y' + a(t)y = 0$, then the general solution to the inhomogeneous equation is

$$y(t) = y_h(t) + y_p(t)\,.$$

Inhomogeneous 2nd order linear ODEs: $y'' + p(t)y' + q(t)y = f(t)$

Big idea 1

Suppose the homogeneous ODE $y'' + p(t)y' + q(t)y = 0$ has solutions $y_1(t)$ and $y_2(t)$. Then

$$C_1 y_1(t) + C_2 y_2(t)$$

is a solution for any constants C_1 and C_2.

Inhomogeneous 2nd order linear ODEs: $y'' + p(t)y' + q(t)y = f(t)$

Big idea 2

The general solution to a 2nd order inhomogeneous ODE is

$$y(t) = y_h(t) + y_p(t) = C_1 y_1(t) + C_1 y_2(t) + y_p(t)\,,$$

where $y_p(t)$ is *any* particular solution and $y_h(t)$ is the general solution to the related homogeneous equation.

How to find a particular solution

Method of undetermined coefficients

1. Guess a solution of the "same type" as the forcing term, $f(t)$.
2. Plug this back in and solve for the unknown coefficients.

Example 1 (exponential forcing term)

Find the general solution to $y'' - 5y' + 4y = e^{3t}$.

Sinusoid forcing term

Example 2

Find the general solution to $y'' + 2y' - 3y = 5\sin 3t$.

Polynomial forcing term

Example 3

Find the general solution to $y'' + 2y' - 3y = 6t^2 + t - 2$.

A problem case

Example 4

Find the general solution to $y'' - 3y' + 2y = e^{2t}$.

Mixed forcing terms

Example 5

Find the general solution to $y'' + 2y' - 3y = 5\sin 3t + 6t^2 + t - 2$.

Mixed forcing terms

Theorem

Suppose:

- $y'' + py' + qy = f(t)$ has solution $y_f(t)$;
- $y'' + py' + qy = g(t)$ has solution $y_g(t)$.

Then $y'' + py' + qy = \alpha f(t) + \beta g(t)$ has solution $\alpha y_f(t) + \beta y_g(t)$.