Lecture 5.6: Convolution

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Introduction

Motivation

Laplace transforms are hard to compute. We like formulas that allow us to compute new ones from old. For example,

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c)\,, \qquad \mathcal{L}\{f(t)+g(t)\} = F(s)+G(s)\,.$$

Question: *Is there a formula for $\mathcal{L}\{f(t)g(t)\}$?*

Next best thing

There *is* a multiplicative formula for the inverse Laplace transform:

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) := \int_0^t f(u)g(t-u)\,du\,.$$

Practice with convolution

Definition

The convolution of $f(t)$ and $g(t)$ is the function $(f * g)(t) := \int_{\mathbb{R}} f(u)g(t-u)\,du$.

Properties

- $f * g = g * f$;
- $f * (g * h) = (f * g) * h$.

Examples

1. Compute $t^2 * t =$

2. Compute $f(t) * 1 =$

Convolutions unrelated to Laplace transforms

Example

Suppose a company dumps radioactive waste at a rate $f(t)$ that decays exponentially with rate constant k. Determine how much waste remains at time t.

Back to (inverse) Laplace transforms

Theorem

If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$, then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) = \int_0^t f(u)g(t-u)\,du\,.$$