# Lecture 6.1: Introduction to Fourier series

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# Introduction

## Motivation

Every "well-behaved" periodic (think: sound wave) function can be decomposed into sine and cosine waves. We'll learn how to do this.

## An analogy

$\mathbb{R}^n$ is a set of vectors. We can freely:

- add & subtract vectors,
- multiply vectors by scalars,
- measure the lengths of vectors; $||\mathbf{v}|| := \sqrt{\mathbf{v} \cdot \mathbf{v}}$,
- measure the angles between vectors; $\measuredangle(\mathbf{v}, \mathbf{w}) := \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}|| \, ||\mathbf{w}||}\right)$,
- project vectors onto unit vectors: $\operatorname{Proj}_{\mathbf{n}} \mathbf{v} := (\mathbf{v} \cdot \mathbf{n})\, \mathbf{n}$.

# An analogy

## Questions

The *standard unit basis vectors* of $\mathbb{R}^2$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $\mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

1. How long is $\mathbf{v}$ in the $x$-direction?
2. How long is $\mathbf{v}$ in the $y$-direction?
3. How long is $\mathbf{v}$ in the "northeast direction"?

# An orthogonal basis of $\mathbb{R}^n$

## Definition

A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is orthonormal if they satisfy: $\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$

# The vector space of periodic functions

Let $\mathcal{P}_{2\pi}$ be the set of $2\pi$-periodic piecewise continuous functions:

$$\mathcal{P}_{2\pi} = \big\{ f \colon \mathbb{R} \to \mathbb{R} \mid f(x+2\pi) = f(x),\ f \text{ is piecewise contin.} \big\}$$

## Definition

The inner product ("generalized dot product") on $\mathcal{P}_{2\pi}$ is defined to be:

$$\langle f(x), g(x) \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x)\, dx\,.$$

# The vector space of periodic functions

## Amazing fact

With respect to our inner product $\langle f, g \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\,dx$ on $\mathcal{P}_{2\pi}$, the set

$$\mathcal{B}_{2\pi} = \left\{ \begin{array}{ccccc} \frac{1}{\sqrt{2}}, & \cos x, & \cos 2x, & \cos 3x, & \dots \\ & \sin x, & \sin 2x, & \sin 3x, & \dots \end{array} \right\}$$

is an orthonormal basis for $\mathcal{P}_{2\pi}$!

# Formula for the Fourier coefficients

**Theorem**

Let $f(x)$ be a piecewise continuous $2\pi$-periodic function. We can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

where

$$a_n = \langle f(x), \cos nx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \langle f(x), \sin nx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

# Periodic functions with other periods

## Remark

Let $f(x)$ be a piecewise continuous $2L$-periodic function. We can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\tfrac{\pi n x}{L}) + b_n \sin(\tfrac{\pi n x}{L}),$$

where

$$a_n = \langle f(x), \cos(\tfrac{\pi n x}{L}) \rangle = \frac{1}{L} \int_{-\pi}^{\pi} f(x) \cos(\tfrac{\pi n x}{L})\, dx$$

$$b_n = \langle f(x), \sin(\tfrac{\pi n x}{L}) \rangle = \frac{1}{L} \int_{-\pi}^{\pi} f(x) \sin(\tfrac{\pi n x}{L})\, dx.$$