Lecture 6.4: Complex Fourier series

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Introduction

A different basis for the set of 2π-periodic functions

Recall that the set

$$\mathcal{B}_{2\pi} = \left\{ \begin{array}{ccccc} \frac{1}{\sqrt{2}}, & \cos x, & \cos 2x, & \cos 3x, & \dots \\ & \sin x, & \sin 2x, & \sin 3x, & \dots \end{array} \right\}$$

is a basis for $\mathcal{P}_{2\pi}$. Here is another basis:

$$\mathcal{B}'_{2\pi} = \left\{ \begin{array}{ccccc} 1, & e^{ix}, & e^{2ix}, & e^{3ix}, & \dots \\ & e^{-ix}, & e^{-2ix}, & e^{-3ix}, & \dots \end{array} \right\}.$$

Moreover, this basis is orthonormal with respect to the inner product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx\,.$$

Formulas for the Fourier coefficients

Complex Fourier series

If $f(x)$ is 2π-periodic, then it can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + c_{-n} e^{-inx})$$

where

$$c_0 = \langle f(x), 1 \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \qquad c_n = \langle f(x), e^{inx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx.$$

From the real to the complex Fourier series

Proposition

The complex Fourier coefficients of $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ are

$$c_n = \frac{a_n - ib_n}{2}, \qquad c_{-n} = \frac{a_n + ib_n}{2}.$$

From the complex to the real Fourier series

Proposition

The real Fourier coefficients of $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ are

$$a_n = c_n + c_{-n}, \qquad b_n = i(c_n - c_{-n}) .$$

Computations

Example 1: square wave

Find the complex Fourier series of $f(x) = \begin{cases} 1, & 0 \leq x < \pi \\ -1, & \pi \leq x < 2\pi \end{cases}$ are

Computations

Example 2

Compute the complex Fourier series of the 2π-periodic extension of the function e^x defined on $(-\pi, \pi]$.