Lecture 7.5: Harmonic functions

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Math 2080, Differential Equations

Higher dimensional PDEs

In 2 dimensions

- Heat equation: $u_t = c^2(u_{xx} + u_{yy})$
- Wave equation: $u_{tt} = c^2(u_{xx} + u_{yy})$

Recall the del operator ∇ from vector calculus:

$$\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right), \qquad \nabla \cdot \nabla = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

Definition

Let $u(x_1, \dots, x_n, t)$ be a function of n spatial variables. The Laplacian of u is

$$\Delta u := \nabla \cdot \nabla\, u = \nabla^2\, u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

In n dimensions

- Heat equation: $u_t = c^2\, \nabla^2 u$
- Wave equation: $u_{tt} = c^2\, \nabla^2 u$

Long-term behavior

Remark

Steady state solutions:

- occur for the heat equation (*heat dissipates*)
- do not occur for the wave equation (*waves propagate*)

Definition

A steady-state solution means $u_t = 0$. Thus, all steady-state solutions satisfy $u_t = c^2 \nabla^2 u = 0$, i.e.,

$$\nabla^2 u = 0 \qquad \Longrightarrow \qquad \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

A function u is harmonic if $\nabla^2 u = 0$.

Properties of harmonic functions

Key properties

- The graphs of harmonic functions ($\nabla^2 f = 0$) are as flat as possible.
- If f is harmonic, then for any closed bounded region R, the function f achieves its minimum and maximum values on the boundary, ∂R.

Examples of harmonic functions