# Lecture 7.7: The two-dimensional heat equation

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# Overview

## Approach

To solve an IVP/BVP problem for the heat equation in two dimensions, $u_t = c^2(u_{xx} + u_{yy})$:

1. Find the steady-state solution $u_{ss}(x, y)$ first, i.e., solve Laplace's equation $\nabla^2 u = 0$ with the same BCs.
2. Solve the related homogeneous equation: set the BCs to zero and keep the same ICs.

Add these two together to get the solution: $u(x, y, t) = u_{ss}(x, y) + u_h(x, y, t)$.

# A homogeneous example

## Example 2a

Solve the following IVP/BVP for the 2D heat equation:

$$u_t = c^2(u_{xx} + u_{yy}), \qquad \begin{aligned} &u(0,y,t) = u(x,0,t) = u(\pi,y,t) = u(x,\pi,t) = 0 \\ &u(x,y,0) = 2\sin x \sin 2y + 3\sin 4x \sin 5y\,. \end{aligned}$$

# Solving the Helmholtz equation

## Example 2a (aside)

We need to solve the following BVP:

$$f_{xx} + f_{yy} = \lambda f, \qquad f(0, y) = f(\pi, y) = f(x, 0) = f(x, \pi) = 0.$$

# A homogeneous example

## Example 2a (cont.)

Solve the following IVP/BVP for the 2D heat equation:

$$u_t = c^2(u_{xx} + u_{yy}), \qquad u(0, y, t) = u(x, 0, t) = u(\pi, y, t) = u(x, \pi, t) = 0$$
$$u(x, y, 0) = 2 \sin x \sin 2y + 3 \sin 4x \sin 5y\,.$$

# An inhomogeneous example

## Example 2b

Solve the following IVP/BVP for the 2D heat equation:

$$u_t = c^2(u_{xx} + u_{yy}), \quad u(0, y, t) = u(x, 0, t) = u(\pi, y, t) = 0, \quad u(x, \pi, t) = x(\pi - x)$$
$$u(x, y, 0) = u_{ss}(x, y) + 2 \sin x \sin 2y + 3 \sin 4x \sin 5y\,.$$