

Math 2080: Differential Equations
Worksheet 4.7: Phase portraits with repeated eigenvalues

NAME:

Consider the system of differential equations: $\begin{cases} x_1' = 4x_1 + x_2, & x_1(0) = -1 \\ x_2' = -1x_1 + 2x_2, & x_2(0) = 1 \end{cases}$

- (a) Write this in matrix form, $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$.
- (b) Knowing that \mathbf{A} has a repeated eigenvalue, $\lambda_{1,2} = 3$, and only one eigenvector, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, write down a solution $\mathbf{x}_1(t)$ to $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
- (c) To find a second solution, assume that $\mathbf{x}_2(t) = te^{3t}\mathbf{v} + e^{3t}\mathbf{w}$. Plug this back into $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and equate coefficients (of te^{3t} and e^{3t}) to get a system of two equations, involving \mathbf{v} , \mathbf{w} , and \mathbf{A} .

- (d) Solve for \mathbf{v} by inspection. Plug this back into the second equation and solve for \mathbf{w} (it will involve a constant, C).
- (e) Using what you got for $\mathbf{v}(t)$ and $\mathbf{w}(t)$, write down a solution $\mathbf{x}_2(t)$ that is not a scalar multiple of \mathbf{x}_1 . (Pick the simplest value of C that works.)
- (f) Write down the general solution, $\mathbf{x}(t)$. As $t \rightarrow \infty$, which of the three terms of $\mathbf{x}(t)$ “grows faster”?
- (g) Sketch the phase portrait. To determine which way the curves “spiral”, compute $\mathbf{x}' = \mathbf{A}\mathbf{x}$ at $\mathbf{x} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$ and see if this velocity vector is pointing upwards or downwards.