Math 2080: Differential Equations
Worksheet 7.3: The transport equation

NAME:

1. The PDE $u_{tt} = c^2 u_{xx}$ is called the *wave equation*. Here it is below written in several different ways.

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = \left(\frac{\partial^2}{\partial t^2} - c\frac{\partial^2}{\partial x^2}\right)u = \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = u_{tt} - c^2 u_{xx} = 0$$

Let $f(x)$ and $g(x)$ be differentiable functions, and define $u(x,t) = f(x+ct) + g(x-ct)$. Compute u_{tt} and u_{xx} and check that $u(x,t)$ solves the wave equation.

2. Consider the following initial value problem for the wave equation:

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = f(x), \quad u_t(x,0) = 0.$$

If $f(x)$ is any differentiable function, then define $u(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct)$.

(a) Let $f(x) = e^{-x^2/2}$. Sketch $u(x,0)$ and $u(x,t)$ for some $t > 0$.

(b) Compute u_t, u_{tt}, and u_{xx} and verify that $u(x,t)$ solves the IVP above.