Math 2080: Differential Equations
Worksheet 7.7: The two-dimensional heat equation

NAME:

1. Consider the following initial/boundary value problem for the heat equation in a square region, where the function $u(x, y, t)$ is defined for $0 \leq x \leq \pi$, $0 \leq y \leq \pi$ and $t \geq 0$.

$$u_t = c^2 \nabla^2 u, \qquad u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0$$
$$u(x,y,0) = 2\sin x \sin y + 5 \sin 2x \sin y.$$

   (a) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of the boundary conditions and the initial condition.

   (b) Assume that there is a solution of the form $u(x, y, t) = f(x, y)g(t)$. Find u_{xx}, u_{yy}, and u_t.

   (c) Plug $u = fg$ back into the PDE and divide both sides by fg (i.e., "separate variables") to get the *eigenvalue problem*. Briefly justify why this quantity must be a constant. Call this constant λ. Write down an ODE for $g(t)$, and a PDE for $f(x, y)$ (the *Helmholz equation*). Include four boundary conditions for $f(x, y)$.

(d) Solve the Helmholz equation and determine λ. You may assume that $f(x, y) = X(x)Y(y)$, then separate variables.

(e) Solve the ODE for $g(t)$.

(f) Find the general solution of the boundary value problem. It will be a superposition (infinite sum) of solutions $u_{nm}(x, y, t) = f_{nm}(x, y)g_{nm}(t)$.

(g) Find the particular solution to the initial value problem that additionally satisfies the initial condition $u(x, y, 0) = 2 \sin x \sin y + 5 \sin 2x \sin y$.

(h) What is the steady-state solution? Give a mathematical *and* intuitive (physical) justification.

2. Consider the following inhomogeneous BVP for the heat equation in a square region.

$$u_t = c^2\,\nabla^2 u, \qquad u(x,0) = u(0,y) = 0, \quad u(x,\pi) = \sin x, \quad u(\pi,y) = \sin 2y.$$

Without knowing the initial conditions, determine the steady-state solution. (*Hint*: If you use your result from the previous worksheet, then almost no actual work is needed on this problem.)