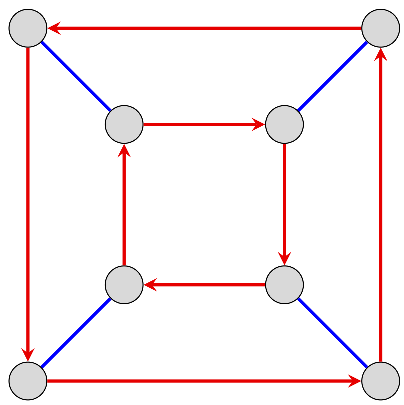
Read the following, which can all be found either in the textbook or on the course website.

- Chapters 6 of *Visual Group Theory* (VGT).
- VGT Exercises 6.6–6.9, 6.12, 6.17–6.20, 6.28–6.30.

Write up solutions to the following exercises.

1. A Cayley diagram and multiplication tables for the dihedral group D_4 are shown below.



	e	r	r^2	r^3	f	rf	r^2f	r^3f
e	e	r	r^2	r^3	f	rf	r^2f	r^3f
r	r	r^2	r^3	e	rf	r^2f	r^3f	f
r^2	r^2	r^3	e	r	r^2f	r^3f	f	rf
r^3	r^3	e	r	r^2	r^3f	f	rf	r^2f
f	f	r^3f	r^2f	rf	e	r^3	r^2	r
rf	rf	f	r^3f	r^2f	r	e	r^3	r^2
r^2f	r^2f	rf	f	r^3f	r^2	r	e	r^3
r^3f	r^3f	r^2f	rf	f	r^3	r^2	r	e

Recall that Section 5.4.4 of VGT describes two algorithms for expressing a group G of order n as a set of permutations in S_n. One algorithm uses the Cayley diagram and the other uses the multiplication table. In this problem, you will explore this a bit further.

(a) Label the vertices of the Cayley diagram from the set $\{1, \dots, 8\}$ and use this to construct a permutation group isomorphic to D_4, and sitting inside S_8.

(b) Label the entries of the multiplication table from the set $\{1, \dots, 8\}$ and use this to construct a permutation group isomorphic to D_4, and sitting inside S_8.

(c) Are the two groups you got in Parts (a) and (b) the same? (The answer will depend on your choice of labeling.) If "yes", then repeat Part (a) with a different labeling to yield a different group. If "no", then repeat Part (a) with a different labeling to yield the group you got in Part (b).

2. Find all subgroups of the following groups, and arrange them in a Hasse diagram, or subgroup lattice. Moreover, label each edge between $K \le H$ with the index, $[H : K]$.

(a) $C_{23} = \langle r \mid r^{23} = 1 \rangle$;

(b) $C_{24} = \langle r \mid r^{24} = 1 \rangle$;

(c) $\mathbb{Z}_3 \times \mathbb{Z}_3 = \{(a, b) \mid a, b \in \{0, 1, 2\}\}$;

(d) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(a, b, c) \mid a, b, c \in \{0, 1\}\}$; (notationally easier to write elements as binary strings, e.g., abc instead of (a, b, c));

(e) $S_3 = \{e, (1\ 2), (2\ 3), (1\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$;

(f) $Q_4 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle$.

3. For each subgroup H of S_4 described below, write out all of its elements and determine what well-known group it is isomorphic to.
   (a) $H = \langle (1\ 2)\,,\ (3\ 4) \rangle$;
   (b) $H = \langle (1\ 2)\,(3\ 4)\,,\ (1\ 3)\,(2\ 4) \rangle$;
   (c) $H = \langle (1\ 2)\,,\ (2\ 3) \rangle$;
   (d) $H = \langle (1\ 2)\,,\ (1\ 3\ 2\ 4) \rangle$;
   (e) $H = \langle (1\ 2\ 3)\,,\ (2\ 3\ 4) \rangle$.

4. Prove the following, algebraically (that is, do not refer to Cayley diagrams):
   (a) If $\mathcal{H}$ is a collection of subgroups of G, then the intersection $\bigcap_{H \in \mathcal{H}} H$ is also a subgroup of G.
   (b) For any subset $S \subseteq G$, the subgroup generated by S is defined as

$$\langle S \rangle := \{ s_1^{e_1} s_2^{e_2} \cdots s_k^{e_k} \mid s_i \in S,\ e_i \in \{-1, 1\} \} \,.$$

   That is, $\langle S \rangle$ consists of all finite "words" that can be written using the elements in S and their inverses. Note that the s_i's need not be distinct. Prove that

$$\langle S \rangle = \bigcap_{S \subseteq H \leq G} H \,,$$

   where the intersection is taken over all subgroups of G that contain S. [*Hint*: One way to prove that $A = B$ is to show that $A \subseteq B$ and $B \subseteq A$.]

5. For a subgroup $H \leq G$ and element $x \in G$, the set $xH := \{ xh \mid h \in H \}$ is a *left coset* of H.
   (a) Prove that if $x \in H$, then $xH = H$. What is the interpretation of this statement in terms of the Cayley diagram?
   (b) Prove that if $b \in aH$, then $aH = bH$.
   (c) Prove that all left cosets have the same size. One way to do this is to prove that for any $x \in G$, the map

$$\varphi \colon H \longrightarrow xH \,, \qquad \varphi \colon h \longmapsto xh$$

   is a bijection.
   (d) Conclude that G is partitioned by the left cosets of H, all of which are equal size.

6. A subgroup H of G is *normal* if $xH = Hx$ for all $x \in G$. Prove that if $[G : H] = 2$, then H is a normal subgroup of G. [*Hint*: Use the results of the previous problem.]