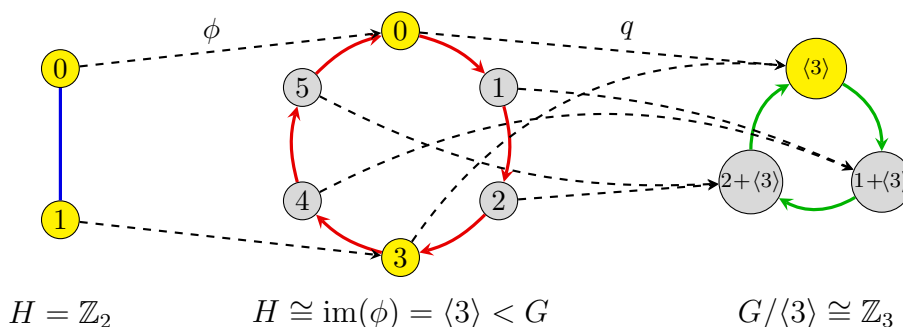


Read the following, which can all be found either in the textbook or on the course website.

- Chapters 8.2–8.5 of *Visual Group Theory* (VGT).
- VGT Exercises 8.19, 8.23, 8/26, 8.27, 8.37(ab), 8.41, 8.43–8.50.

Write up solutions to the following exercises.

1. Let \mathbb{Q} be the group of rational numbers under addition, \mathbb{Q}^* be the group of non-zero rational numbers under multiplication, and let \mathbb{Q}^+ be the group of positive rational numbers under multiplication.
 - (a) Show that $\mathbb{Q}^* \cong \mathbb{Q}^+ \times C_2$. [Hint: Recall that $C_2 = \{e^{0\pi i}, e^{\pi i}\} = \{1, -1\}$.]
 - (b) Describe the quotient groups $\mathbb{Q}/\langle 1 \rangle$ and $\mathbb{Q}^*/\langle -1 \rangle$. In particular, what do the elements (cosets) look like?
 - (c) Use the Fundamental Homomorphism Theorem to prove that $\mathbb{Q}^*/\langle -1 \rangle \cong \mathbb{Q}^+$.
2. For Parts (a)–(d), a group G is given together with a normal subgroup H . Illustrate the embedding $\phi: H \rightarrow G$, and the quotient map $q: G \rightarrow G/H$, chained together so that $\text{im}(\phi) = \ker(q)$. An example for $G = \mathbb{Z}_6$ and $H = \mathbb{Z}_2$ is shown below:



- (a) $G = \mathbb{Z}_6$, $H = \mathbb{Z}_3$,
- (b) $G = D_3$, $H = C_3$,
- (c) $G = A_4$, $H = V_4$,
- (d) $G = S_n$, $H = A_n$ [don't draw the actual Cayley graphs for this one, just the maps].

Now, answer each of the following questions about each of your answers to Parts (a)–(d).

- (e) What map θ into H would satisfy the equation $\text{im } \theta = \ker \phi$? Choose one with the smallest possible domain.
- (f) What map θ' from G/H would satisfy the equation $\text{im}(q) = \ker(\theta')$? Choose one with the smallest possible codomain.
- (g) Add the two maps θ and θ' to your illustration.
- (h) The new chain of four homomorphisms is called a *short exact sequence*. It is one way to use homomorphisms to illustrate quotients, and it shows a connection between embeddings and quotient maps. Given a normal subgroup $H \triangleleft G$, show how to create a short exact sequence involving G and H .

3. Let A and B be normal subgroups of G . In this problem, you will prove the *Diamond Isomorphism Theorem*.

(a) Prove that the set $AB := \{ab : a \in A, b \in B\}$ is a subgroup of G .

(b) Prove that $B \triangleleft AB$ and $A \cap B \triangleleft A$.

(c) Prove that $A/(A \cap B) \cong AB/B$. [*Hint*: Construct a homomorphism $\phi: A \rightarrow AB/B$ that has kernel $A \cap B$, then apply the FHT.]

(d) Draw a diagram, or lattice, of G and its subgroups AB , A , B , and $A \cap B$. Interpret the result in Part (c) in terms of this diagram.

4. For each part below, consider the group $G = \langle A, B \rangle$ generated by the two matrices shown. Assume that matrix multiplication is the binary operation, and $i = \sqrt{-1}$. To what common group is G isomorphic? Write down an explicit isomorphism (you only need to define it for the generators), and a group presentation for G .

$$(a) \quad A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$(b) \quad A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

$$(c) \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

5. In this exercise, you will prove that if A and B are normal subgroups of G , and $AB = G$, then

$$G/(A \cap B) \cong (G/A) \times (G/B).$$

(a) Consider the following map:

$$\phi: AB \longrightarrow (G/A) \times (G/B), \quad \phi(g) = (gA, gB).$$

Show that ϕ is a homomorphism.

(b) Show that ϕ is surjective. That is, given any (g_1A, g_2B) , show that there is some $g = ab \in AB$ such that $\phi(g) = (g_1A, g_2B)$. [*Hint*: Try $g = a_2b_1$.]

(c) Find $\ker(\phi)$ [you need to prove your answer is correct] and then apply the Fundamental Homomorphism Theorem.

6. For each order given below, list all abelian groups of that order by writing each one as a product of cyclic groups of prime power order. Additionally, write each one as a product of cyclic groups organized by “elementary divisors.”

(a) 8

(b) 54

(c) 400

(d) p^2q , where p and q are distinct primes.