Read the following, which can all be found either in the textbook or on the course website.

- Chapters 9.2–9.4 of *Visual Group Theory* (VGT).
- VGT Exercises 9.1, 9.4, 9.12, 9.14, 9.15, 9.19–9.27.

Write up solutions to the following exercises.

1. Let S be the following set of 7 "binary squares":

$$S = \left\{ \begin{array}{|cc|}\hline 0 & 0 \\ 0 & 0 \\ \hline\end{array}, \begin{array}{|cc|}\hline 0 & 1 \\ 1 & 0 \\ \hline\end{array}, \begin{array}{|cc|}\hline 1 & 0 \\ 0 & 1 \\ \hline\end{array}, \begin{array}{|cc|}\hline 1 & 1 \\ 0 & 0 \\ \hline\end{array}, \begin{array}{|cc|}\hline 0 & 1 \\ 0 & 1 \\ \hline\end{array}, \begin{array}{|cc|}\hline 0 & 0 \\ 1 & 1 \\ \hline\end{array}, \begin{array}{|cc|}\hline 1 & 0 \\ 1 & 0 \\ \hline\end{array} \right\}$$

   (a) Consider the (right) action of the group $G = V_4 = \langle v, h \rangle$ on S, where $\phi(v)$ reflects each square vertically, and $\phi(h)$ reflects each square horizontally. Draw an action diagram and compute the stabilizer of each element.

   (b) Consider the (right) action of the group $G = C_4 = \langle r \mid r^4 = e \rangle$ on S, where $\phi(r)$ rotates each square 90° clockwise. Draw an action diagram and compute the stabilizer of each element.

   (c) Suppose a group G of size 15 acts on S. Prove that there must be a fixed point.

2. Let $G = S_4$ act on itself by conjugation via the homomorphism

$$\phi \colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg.$$

   (a) How many orbits are there? Describe them as specifically as you can.

   (b) Find the orbit and the stabilizer of the following elements:

      i. e
      ii. $(1\ 2)$
      iii. $(1\ 2\ 3)$
      iv. $(1\ 2\ 3\ 4)$

3. A *p-group* is a group of order p^k for some integer k. Recall that the *center* of a group G is the set of all elements that commute with everything:

$$\begin{aligned} Z(G) &= \{z \in G \mid gz = zg,\ \forall g \in G\} \\ &= \{z \in G \mid g^{-1}zg = z,\ \forall g \in G\}\,. \end{aligned}$$

   Finally, a group G is *simple* if its only normal subgroups are G and $\langle e \rangle$.

   (a) Let G act on itself by conjugation via the homomorphism

$$\phi \colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg.$$

   Prove that $\mathrm{Fix}(\phi) = Z(G)$.

(b) Prove that if G is a p-group, then $|Z(G)| > 1$. [*Hint*: Revisit the Class Equation.]

(c) Use the result of the previous part to classify all simple p-groups.

4. Let G be an unknown group of order 8. By the First Sylow Theorem, G must contain a subgroup H of order 4.

(a) If all subgroups of G of order 4 are isomorphic to V_4, then what group must G be? Completely justify your answer.

(b) Next, suppose that G has a subgroup $H \cong C_4$. Then G has a Cayley diagram like one of the following:

Find all possibilities for finishing the Cayley diagram.

(c) Label each completed Cayley diagram by isomorphism type. Justify your answer.

(d) Make a complete list of all groups of order 8, up to isomorphism.

5. Recall that a group G is called *simple* if its only normal subgroups are G and $\{e\}$.

(a) Show that there is no simple group of order $45 = 3^2 \cdot 5$.

(b) Show that there is no simple group of order pq, where $p < q$ and are both prime.

(c) Show that there is no simple group of order $12 = 2^2 \cdot 3$.

(d) Show that there is no simple group of order $56 = 2^3 \cdot 7$.

(e) Show that there is no simple group of order $108 = 2^2 \cdot 3^3$.