Read the following, which can all be found either in the textbook or on the course website.

- Chapters 9.2–9.4 of Visual Group Theory (VGT).
- VGT Exercises 9.1, 9.4, 9.12, 9.14, 9.15, 9.19–9.27.

Write up solutions to the following exercises.

1. Let S be the following set of 7 "binary squares":

$$S = \left\{ \begin{array}{cccc} 0 & 0 \\ 0 & 0 \end{array}, \begin{array}{cccc} 0 & 1 \\ 1 & 0 \end{array}, \begin{array}{cccc} 1 & 0 \\ 0 & 1 \end{array}, \begin{array}{ccccc} 1 & 1 & 0 \\ 0 & 1 \end{array}, \begin{array}{ccccc} 0 & 1 \\ 0 & 1 \end{array}, \begin{array}{ccccc} 0 & 0 \\ 0 & 1 \end{array}, \begin{array}{ccccc} 1 & 0 \\ 1 & 0 \end{array} \right\}$$

- (a) Consider the (right) action of the group  $G = V_4 = \langle v, h \rangle$  on S, where  $\phi(v)$  reflects each square vertically, and  $\phi(h)$  reflects each square horizontally. Draw an action diagram and compute the stabilizer of each element.
- (b) Consider the (right) action of the group  $G = C_4 = \langle r \mid r^4 = e \rangle$  on S, where  $\phi(r)$  rotates each square 90° clockwise. Draw an action diagram and compute the stabilizer of each element.
- (c) Suppose a group G of size 15 acts on S. Prove that there must be a fixed point.
- 2. Let  $G = S_4$  act on itself by conjugation via the homomorphism

$$\phi \colon G \longrightarrow \operatorname{Perm}(S)$$
,  $\phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg$ .

- (a) How many orbits are there? Describe them as specifically as you can.
- (b) Find the orbit and the stabilizer of the following elements:
  - i. *e*
  - ii. (12)
  - iii. (1 2 3)
  - iv. (1 2 3 4)
- 3. A p-group is a group of order  $p^k$  for some integer k. Recall that the *center* of a group G is the set of all elements that commute with everything:

$$Z(G) = \{ z \in G \mid gz = zg, \ \forall g \in G \}$$
$$= \{ z \in G \mid g^{-1}zg = z, \ \forall g \in G \}.$$

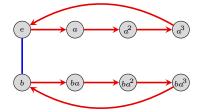
Finally, a group G is *simple* if its only normal subgroups are G and  $\langle e \rangle$ .

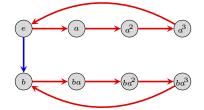
(a) Let G act on itself by conjugation via the homomorphism

$$\phi \colon G \longrightarrow \operatorname{Perm}(S) \,, \qquad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg.$$

Prove that  $Fix(\phi) = Z(G)$ .

- (b) Prove that if G is a p-group, then |Z(G)| > 1. [Hint: Revisit the Class Equation.]
- (c) Use the result of the previous part to classify all simple p-groups.
- 4. Let G be an unknown group of order 8. By the First Sylow Theorem, G must contain a subgroup H of order 4.
  - (a) If all subgroups of G of order 4 are isomorphic to  $V_4$ , then what group must G be? Completely justify your answer.
  - (b) Next, suppose that G has a subgroup  $H \cong C_4$ . Then G has a Cayley diagram like one of the following:





Find all possibilities for finishing the Cayley diagram.

- (c) Label each completed Cayley diagram by isomorphism type. Justify your answer.
- (d) Make a complete list of all groups of order 8, up to isomorphism.
- 5. Recall that a group G is called *simple* if its only normal subgroups are G and  $\{e\}$ .
  - (a) Show that there is no simple group of order  $45 = 3^2 \cdot 5$ .
  - (b) Show that there is no simple group of order pq, where p < q and are both prime.
  - (c) Show that there is no simple group of order  $12 = 2^2 \cdot 3$ .
  - (d) Show that there is no simple group of order  $56 = 2^3 \cdot 7$ .
  - (e) Show that there is no simple group of order  $108 = 2^2 \cdot 3^3$ .