

Math 4120/6120, Summer I 2016 (online)

Study guide: Midterm 1.

Note: This is just a guide, not an all-inclusive list.

Definitions to memorize.

- (1) A *group* G . (The “official” definition.)
- (2) A *left coset* xH of a subgroup $H \leq G$.
- (3) A *normal subgroup* $H \triangleleft G$.
- (4) The *index* $[G : H]$ of a subgroup $H \leq G$.
- (5) The *direct product* $A \times B$ of two groups A and B .
- (6) The *quotient* G/H of a group G by a normal subgroup $H \triangleleft G$.
- (7) What it means for a map $f: G/N \rightarrow H$ to be *well-defined*.
- (8) The *normalizer* $N_G(H)$ of a subgroup $H \triangleleft G$.
- (9) The *conjugacy class* $\text{cl}_G(x)$ of an element $x \in G$.
- (10) The *center* $Z(G)$ of a group.
- (11) A *homomorphism* ϕ from a group G to a group H .
- (12) An *isomorphism* ϕ from a group G to a group H .
- (13) The *kernel* $\ker \phi$ of a homomorphism $\phi: G \rightarrow H$.

Useful facts and techniques.

- (1) Two different ways to show that a subset $H \subseteq G$ is a subgroup.
- (2) Three different ways to show that a subgroup $H \leq G$ is normal.
- (3) Two elements in S_n are conjugate iff they have the same cycle type.

Proofs to learn.

- (1) Prove that the identity element of a group is unique.
- (2) Prove that every element in a group has a unique inverse.
- (3) Prove that if $\{H_\alpha \mid \alpha \in I\}$ is a collection of subgroups, then $\bigcap_{\alpha \in I} H_\alpha$ is a subgroup.
- (4) Prove that $xH = H$ if and only if $x \in H$.
- (5) Prove that if $[G : H] = 2$, then $H \triangleleft G$.
- (6) Prove that if $K \leq H \leq G$ and $K \triangleleft G$, then $K \triangleleft H$.
- (7) Prove that the center $Z(G) = \{z \in G \mid gz = zg, \forall g \in G\}$ is a subgroup of G and that it is normal.
- (8) Let $H \triangleleft G$. Prove that multiplication of cosets is well-defined: if $a_1H = a_2H$ and $b_1H = b_2H$, then $a_1H \cdot b_1H = a_2H \cdot b_2H$. Additionally, show that G/H is a group under this binary operation.
- (9) Prove that if G is abelian and $H \leq G$, then G/H is abelian.
- (10) Prove that the normalizer $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$ is a subgroup of G .
- (11) Prove that $\text{cl}_G(x) = \{x\}$ if and only if $x \in Z(G)$.
- (12) Let $\phi: G \rightarrow H$ be a homomorphism. Prove that $\phi(1_G) = 1_H$, where 1_G and 1_H are the identity elements of G and H , respectively. Additionally, prove that $\phi(g^{-1}) = \phi(g)^{-1}$ for all $g \in G$.
- (13) Let $\phi: G \rightarrow H$ be a homomorphism. Prove that $\ker \phi := \{k \in G \mid \phi(k) = 1_H\}$ is a subgroup of G , and that it is normal.
- (14) Prove that $A \times B \cong B \times A$.
- (15) Prove that if $H \leq G$, then $xHx^{-1} \cong H$ for any $x \in G$.

Study guide: Midterm 2.

Definitions to memorize.

- (1) The *commutator subgroup* G' of a group G , and the *abelianization* G/G' .
- (2) A *group action* of G on a set S .
- (3) The *orbit* of an element $s \in S$.
- (4) The *stabilizer* of an element $s \in S$.
- (5) The *fixed points* of a group action.
- (6) A *p-group*, and a *Sylow p-subgroup* of a group G .
- (7) A *field* F .
- (8) A *field automorphism* of F .
- (9) The *degree* $[E : F]$ of a field extension E of F .
- (10) What it means for a number $\alpha \notin \mathbb{Q}$ to be *algebraic*.
- (11) What it means for a field to be *algebraically closed*.
- (12) The *Galois group* of a field extension, and of a polynomial.

Useful facts and techniques.

- (1) $\mathbb{Z}_n \times \mathbb{Z}_m$ iff $\gcd(n, m) = 1$.
- (2) Learn to classify all finite abelian groups of a fixed order.
- (3) There are two ways to prove that $G/N \cong H$: Either construct a map $G/N \rightarrow H$ and prove it is a well-defined bijective homomorphism, or construct a map $\phi: G \rightarrow H$ and prove it is an onto homomorphism with $\ker \phi = N$.
- (4) Learn the statement of the Correspondence Theorem: There is a 1–1 correspondence between subgroup of G/N and subgroups of G that contain N . Moreover, every subgroup of G/N is of the form H/N for some $N \leq H \leq G$.
- (5) Learn how to identify the commutator subgroup of G just from the subgroup lattice.
- (6) $\text{Aut}(\mathbb{Z}_n) \cong U_n$.
- (7) The orbit-stabilizer theorem: If G acts on S , then $|G| = |\text{Orb}(s)| \cdot |\text{Stab}(s)|$ for any $s \in S$.
- (8) Learn what the orbits, stabilizers, and fixed points are of the following actions:
 - (i) G acting on itself by right multiplication.
 - (ii) G acting on itself by conjugation.
 - (iii) G acting on its subgroups by conjugation.
 - (iv) G acting on its right cosets by right multiplication.
- (9) Learn how to use the 3rd Sylow theorem to show that a group of a certain order is simple. (Usually, by showing that $n_p = 1$ for some prime p .)
- (10) Use Eisenstein's criterion to show that a particular polynomial is irreducible.

Proofs to learn.

- (1) Prove there is no embedding $\varphi: \mathbb{Z}_n \rightarrow \mathbb{Z}$.
- (2) Prove that if $\varphi: G \rightarrow H$ is a homomorphism and $N \triangleleft H$, then $\varphi^{-1}(N)$ is a normal subgroup of G .
- (3) If $H \leq G$ is the only subgroup of G of order $|H|$, then H must be normal.
- (4) The FHT: If $\varphi: G \rightarrow H$ is a homomorphism, then $G/\ker \varphi \cong \text{im } \varphi$.
- (5) The Diamond Isomorphism Theorem: If $A, B \triangleleft G$, then $AB \leq G$, $B \triangleleft AB$, $(A \cap B) \triangleleft A$, and $AB/B \cong A/(A \cap B)$.
- (6) Show that $\mathbb{Q}^* \cong \mathbb{Q}^+ \times C_2$ and $\mathbb{Q}^*/\langle -1 \rangle \cong \mathbb{Q}^+$, where \mathbb{Q}^* is the nonzero rationals under multiplication, and $\mathbb{Q}^+ \leq \mathbb{Q}^*$ is the subgroup of positive rationals.
- (7) Prove that G is abelian iff its commutator subgroup $G' = \{e\}$.
- (8) Prove that G/G' is abelian.
- (9) Show that if G acts on S , then $\text{Stab}(s)$ is a subgroup of G , for any $s \in S$.
- (10) Prove that if G is a p -group, then $|Z(G)| > 1$. (Use the class equation.)
- (11) Use Galois theory to prove that $\sqrt{2}$ is irrational.

Study guide: Final exam.

Note: This is *in addition*, not instead, of the Midterm 1 and 2 material.

Definitions to memorize.

- (1) The *degree* of a field extension $[E : F]$.
- (2) The *minimal polynomial* of a number $r \notin F$.
- (3) What it means for an extension field E of F to be *normal*.
- (4) What it means for group G to be *solvable*.
- (5) A *ring* R .
- (6) A *unit*, and a *zero divisor* of a ring.
- (7) Types of rings: integral domain, division ring, principle ideal domain (PID), unique factorization domain (UFD), Euclidean domain.
- (8) An *ideal* of a ring R (left, right, and two-sided).
- (9) The *quotient ring* R/I for some two-sided ideal I , and how to multiply elements.
- (10) A *homomorphism* ϕ from a ring R to a ring S .
- (11) A *maximal ideal* M of a ring R .
- (12) A *prime ideal* P of a ring R .

Useful facts and techniques.

- (1) The degree of an extension $\mathbb{Q}(r)$ is the degree of the minimal polynomial of r .
- (2) The Galois group of $f(x)$ acts on its n roots, and so $\text{Gal}(f(x)) \leq S_n$. If f is irreducible, then this action has only one orbit.
- (3) $|\text{Gal}(f(x))| = [K : \mathbb{Q}]$, where K is the splitting field of $f(x)$.
- (4) Know the statement of the Fundamental Theorem of Galois theory.
- (5) Know the Galois groups of the following field extensions and be able to describe the explicit automorphisms: $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, $\mathbb{Q}(\sqrt[3]{2}, \sqrt{3}i)$, $\mathbb{Q}(\sqrt[4]{2}, i)$, and $\mathbb{Q}(\zeta_n)$, where ζ_n is an n^{th} root of unity.
- (6) Be able to construct the subfield lattices of the above fields, and demonstrate the Galois correspondence with subgroups of $\text{Gal}(f(x))$.
- (7) Know the Galois groups of the following polynomials: $f(x) = x^2 - 2$, $f(x) = (x^2 - 2)(x^2 - 3)$, $f(x) = x^3 - 2$, $f(x) = x^4 - 2$, $f(x) = x^n - 1$.
- (8) Summarize in a few sentences how to construct a degree-5 polynomial that is not solvable by radicals.
- (9) Know examples of each of the following types of rings: integral domain, division ring, principle ideal domain (PID), unique factorization domain (UFD), Euclidean domain.
- (10) Know examples of both maximal ideals and prime ideals.
- (11) Learn how to construct a finite field \mathbb{F}_q of order $q = p^k$.
- (12) Know the statements of the fundamental homomorphism theorem and the correspondence theorem for rings and how to apply them.

Proofs to learn.

- (1) If an ideal I of R contains a unit, then $I = R$.
- (2) The FHT for rings: if $\phi: R \rightarrow S$ is a ring homomorphism, then $\ker \phi$ is an ideal of R and $R/\ker \phi \cong \text{im } \phi$.
- (3) The following are equivalent: (i) I is a maximal ideal, (ii) R/I is simple, (iii) R/I is a field.
- (4) An ideal P is prime iff R/P is an integral domain.
- (5) A ring R is an integral domain iff 0 is a prime ideal.