Lecture 3.4: Direct products

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Math 4120, Modern Algebra

Overview

Previously, we looked for smaller groups lurking inside a group.

Exploring the subgroups of a group gives us insight into the its internal structure.

The next two lectures are about the following topics:

1. direct products: a method for making *larger* groups from smaller groups.
2. quotients: a method for making *smaller* groups from larger groups.

Before we begin, we'll note that we can *always* form a direct product of two groups.

In constrast, we cannot always take the quotient of two groups. In fact, quotients are restricted to some pretty specific circumstances, as we shall see.

Direct products, algebraically

It is easiest to think of direct product of groups algebraically, rather than visually.

If A and B are groups, there is a natural group structure on the set

$$A \times B = \{(a,b) \mid a \in A,\ b \in B\}.$$

Definition

The **direct product** of groups A and B consists of the set $A \times B$, and the group operation is done component-wise: if $(a,b),(c,d) \in A \times B$, then

$$(a,b) * (c,d) = (ac, bd).$$

We call A and B the factors of the direct product.

Note that the binary operations on A and B could be different. One might be $*$ and the other $+$.

For example, in $D_3 \times \mathbb{Z}_4$:

$$(r^2,1) * (fr,3) = (r^2 fr, 1+3) = (rf, 0).$$

These elements do *not* commute:

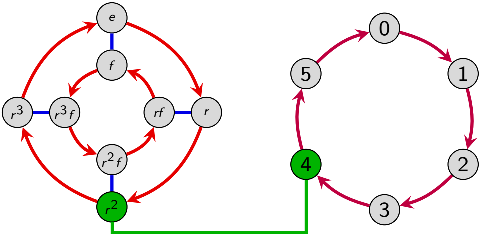
$$(fr,3) * (r^2,1) = (fr^3, 3+1) = (f,0).$$

Direct products, visually

Here's one way to think of the direct product of two cyclic groups, say $\mathbb{Z}_n \times \mathbb{Z}_m$: Imagine a slot machine with two wheels, one with n spaces (numbered 0 through $n-1$) and the other with m spaces (numbered 0 through $m-1$).

The actions are: spin one or both of the wheels. Each action can be labeled by where we end up on each wheel, say (i, j).

Here is an example for a more general case: the element $(r^2, 4)$ in $D_4 \times \mathbb{Z}_6$.



Key idea

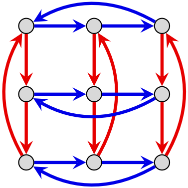
The direct product of two groups joins them so they act independently of each other.

Cayley diagrams of direct products

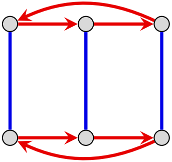
Remark

Just because a group is not written with $\times$ doesn't mean that there isn't some hidden direct product structure lurking. For example, V_4 is really just $C_2 \times C_2$.

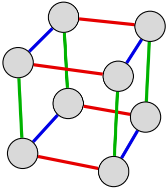
Here are some examples of direct products:



$C_3 \times C_3$



$C_3 \times C_2$



$C_2 \times C_2 \times C_2$

Even more surprising, the group $C_3 \times C_2$ is actually isomorphic to the cyclic group C_6!

Indeed, the Cayley diagram for C_6 using generators r^2 and r^3 is the same as the Cayley diagram for $C_3 \times C_2$ above.

We'll understand this better later in the class when we study homomorphisms. For now, we will focus our attention on direct products.

Cayley diagrams of direct products

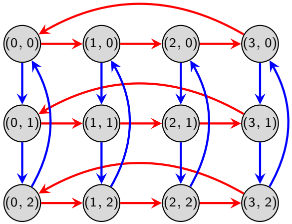
Let e_A be the identity of A and e_B the identity of B.

Given a Cayley diagram of A with generators $a_1, \ldots, a_k$, and a Cayley diagram of B with generators $b_1, \ldots, b_\ell$, we can create a Cayley diagram for $A \times B$ as follows:

- Vertex set: $\{(a, b) \mid a \in A,\ b \in B\}$.
- Generators: $(a_1, e_b), \ldots, (a_k, e_b)$ and $(e_a, b_1), \ldots, (e_a, b_\ell)$.

Frequently it is helpful to arrange the vertices in a rectangular grid.

For example, here is a Cayley diagram for the group $\mathbb{Z}_4 \times \mathbb{Z}_3$:



What are the subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_3$? There are six (did you find them all?), they are:

$$\mathbb{Z}_4 \times \mathbb{Z}_3, \qquad \{0\} \times \{0\}, \qquad \{0\} \times \mathbb{Z}_3, \qquad \mathbb{Z}_4 \times \{0\}, \qquad \mathbb{Z}_2 \times \mathbb{Z}_3, \qquad \mathbb{Z}_2 \times \{0\}.$$

Subgroups of direct products

Remark

If $H \leq A$, and $K \leq B$, then $H \times K$ is a subgroup of $A \times B$.

For $\mathbb{Z}_4 \times \mathbb{Z}_3$, *all* subgroups had this form. However, this is not always true.

For example, consider the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, which is really just V_4. Since $\mathbb{Z}_2$ has two subgroups, the following four sets are subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$:

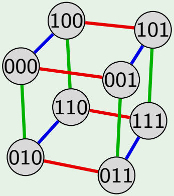
$$\mathbb{Z}_2 \times \mathbb{Z}_2, \qquad \{0\} \times \{0\}, \qquad \mathbb{Z}_2 \times \{0\} = \langle (1,0) \rangle, \qquad \{0\} \times \mathbb{Z}_2 = \langle (0,1) \rangle.$$

However, one subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is missing from this list: $\langle (1,1) \rangle = \{(0,0), (1,1)\}$.

Exercise

What are the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$?

Here is a Cayley diagram, writing the elements of the product as abc rather than (a, b, c).



Hint: There are 16 subgroups!

Direct products, visually

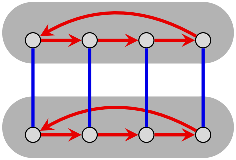
It's not needed, but one can construct the Cayley diagram of a direct product using the following "inflation" method.

Inflation algorithm

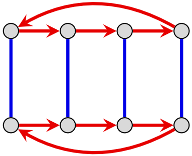
To make a Cayley diagram of $A \times B$ from the Cayley diagrams of A and B:

1. Begin with the Cayley diagram for A.
2. Inflate each node, and place in it a copy of the Cayley diagram for B. (Use different colors for the two Cayley diagrams.)
3. Remove the (inflated) nodes of A while using the arrows of A to connect corresponding nodes from each copy of B. That is, remove the A diagram but treat its arrows as a blueprint for how to connect corresponding nodes in the copies of B.

Cyclic group $\mathbb{Z}_2$



each node contains a copy of $\mathbb{Z}_4$



direct product group $\mathbb{Z}_4 \times \mathbb{Z}_2$

Properties of direct products

Recall the following definition from the previous lecture.

Definition

A subgroup $H < G$ is normal if $xH = Hx$ for all $x \in G$. We denote this by $H \lhd G$.

Assuming A and B are not trivial, the direct product $A \times B$ has *at least* four normal subgroups:

$$\{e_A\} \times \{e_B\}\,, \qquad A \times \{e_B\}\,, \qquad \{e_A\} \times B\,, \qquad A \times B\,.$$

Sometimes we "abuse notation" and write $A \lhd A \times B$ and $B \lhd A \times B$ for the middle two. (Technically, A and B are not even subsets of $A \times B$.)

Here's another observation: "A-arrows" are independent of "B-arrows."

Observation

In a Cayley diagram for $A \times B$, following "A-arrows" neither impacts or is impacted by the location in group B.

Algebraically, this is just saying that $(a, e_b) * (e_a, b) = (a, b) = (e_a, b) * (a, e_b)$.

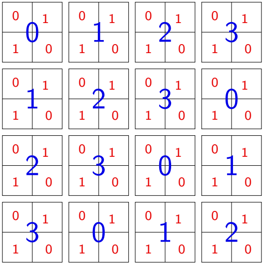
Multiplication tables of direct products

Direct products can also be visualized using multiplication tables.

The general process should be clear after seeing the following example; constructing the table for the group $\mathbb{Z}_4 \times \mathbb{Z}_2$:

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

multiplication table for the group $\mathbb{Z}_4$



inflate each cell to contain a copy of the multiplication table of $\mathbb{Z}_2$

(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)	(3,0)	(3,1)
(0,1)	(0,0)	(1,1)	(1,0)	(2,1)	(2,0)	(3,1)	(3,0)
(1,0)	(1,1)	(2,0)	(2,1)	(3,0)	(3,1)	(0,0)	(0,1)
(1,1)	(1,0)	(2,1)	(2,0)	(3,1)	(3,0)	(0,1)	(0,0)
(2,0)	(2,1)	(3,0)	(3,1)	(0,0)	(0,1)	(1,0)	(1,1)
(2,1)	(2,0)	(3,1)	(3,0)	(0,1)	(0,0)	(1,1)	(1,0)
(3,0)	(3,1)	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)
(3,1)	(3,0)	(0,1)	(0,0)	(1,1)	(1,0)	(2,1)	(2,0)

join the little tables and element names to form the direct product table for $\mathbb{Z}_4 \times \mathbb{Z}_2$