Lecture 4.5: The isomorphism theorems

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The Isomorphism Theorems

The Fundamental Homomorphism Theorem (FHT) is the first of four basic theorems about homomorphism and their structure.

These are commonly called "The Isomorphism Theorems":

- First Isomorphism Theorem: "Fundamental Homomorphism Theorem"
- Second Isomorphism Theorem: "Diamond Isomorphism Theorem"
- Third Isomorphism Theorem: "Freshman Theorem"
- Fourth Isomorphism Theorem: "Correspondence Theorem"

All of these theorems have analogues in other algebraic structures: rings, vector spaces, modules, and Lie algebras, to name a few.

In this lecture, we will summarize the last three isomorphism theorems and provide visual pictures for each.

We will prove one, outline the proof of another (homework!), and encourage you to try the (very straightforward) proofs of the multiple parts of the last one.

Finally, we will introduce the concepts of a commutator and commutator subgroup, whose quotient yields the abelianation of a group.

The Second Isomorphism Theorem

Diamond isomorphism theorem

Let $H \leq G$, and $N \lhd G$. Then

(i) The product $HN = \{hn \mid h \in H,\ n \in N\}$ is a subgroup of G.

(ii) The intersection $H \cap N$ is a *normal* subgroup of G.

(iii) The following quotient groups are isomorphic:

$$HN/N \cong H/(H \cap N)$$

$$\begin{tikzcd}[column sep=small]
 & G \arrow[d, dash] & \\
 & HN \arrow[dl, dash] \arrow[dr, dash] & \\
H \arrow[dr, dash] & & N \arrow[dl, dash] \\
 & H \cap N &
\end{tikzcd}$$

Proof (sketch)

Define the following map

$$\phi \colon H \longrightarrow HN/N\,, \qquad \phi \colon h \longmapsto hN\,.$$

If we can show:

1. ϕ is a homomorphism,
2. ϕ is surjective (onto),
3. $\operatorname{Ker}\phi = H \cap N$,

then the result will follow *immediately* from the FHT. The details are left as HW.

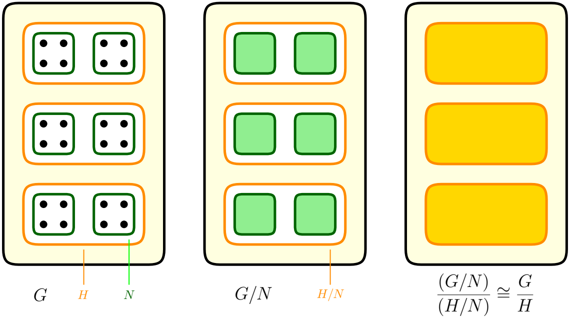
The Third Isomorphism Theorem

Freshman theorem

Consider a chain $N \leq H \leq G$ of normal subgroups of G. Then

1. The quotient H/N is a normal subgroup of G/N;
2. The following quotients are isomorphic:

$$(G/N)/(H/N) \cong G/H\,.$$



(Thanks to Zach Teitler of Boise State for the concept and graphic!)

The Third Isomorphism Theorem

Freshman theorem

Consider a chain $N \leq H \leq G$ of normal subgroups of G. Then $H/N \lhd G/N$ and $(G/N)/(H/N) \cong G/H$.

Proof

It is easy to show that $H/N \lhd G/N$ (exercise). Define the map

$$\varphi\colon G/N \longrightarrow G/H, \qquad \varphi\colon gN \longmapsto gH.$$

- *Show φ is well-defined*: Suppose $g_1N = g_2N$. Then $g_1 = g_2n$ for some $n \in N$. But $n \in H$ because $N \leq H$. Thus, $g_1H = g_2H$, i.e., $\varphi(g_1N) = \varphi(g_2N)$. ✓
- *φ is clearly onto and a homomorphism*. ✓
- *Apply the FHT*:

$$\begin{aligned} \operatorname{Ker}\varphi &= \{gN \in G/N \mid \varphi(gN) = H\} \\ &= \{gN \in G/N \mid gH = H\} \\ &= \{gN \in G/N \mid g \in H\} = H/N \end{aligned}$$

By the FHT, $(G/N)/\operatorname{Ker}\varphi = (G/N)/(H/N) \cong \operatorname{Im}\varphi = G/H$. $\square$

The Fourth Isomorphism Theorem

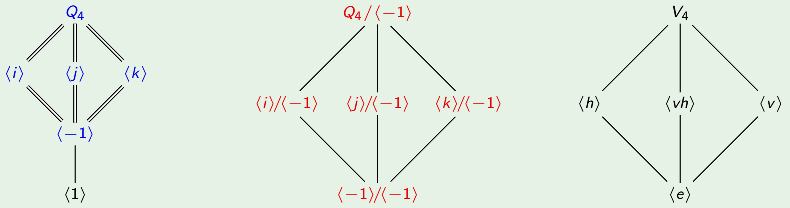
The full statement is a bit technical, so here we just state it informally.

Correspondence theorem

Let $N \lhd G$. There is a 1–1 correspondence between subgroups of G/N and subgroups of G that contain N. In particular, every subgroup of G/N has the form $\overline{A} := A/N$ for some A satisfying $N \leq A \leq G$.

This means that the corresponding subgroup lattices are identical in structure.

Example



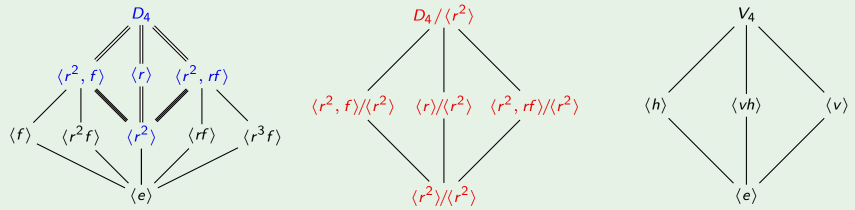
The quotient $Q_4/\langle -1 \rangle$ is isomorphic to V_4. The subgroup lattices can be visualized by "collapsing" $\langle -1 \rangle$ to the identity.

Correspondence theorem (formally)

Let $N \lhd G$. Then there is a bijection from the subgroups of G/N and subgroups of G that contain N. In particular, every subgroup of G/N has the form $\overline{A} := A/N$ for some A satisfying $N \leq A \leq G$. Moreover, if $A, B \leq G$, then

1. $A \leq B$ if and only if $\overline{A} \leq \overline{B}$,
2. If $A \leq B$, then $[B : A] = [\overline{B} : \overline{A}]$,
3. $\overline{\langle A, B\rangle} = \langle \overline{A}, \overline{B}\rangle$,
4. $\overline{A \cap B} = \overline{A} \cap \overline{B}$,
5. $A \lhd G$ if and only if $\overline{A} \lhd \overline{G}$.

Example

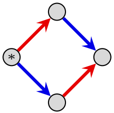


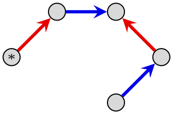
Application: commutator subgroups and abelianizations

We've seen how to divide $\mathbb{Z}$ by $\langle 12\rangle$, thereby "forcing" all multiples of 12 to be zero. This is one way to construct the integers modulo 12: $\mathbb{Z}_{12} \cong \mathbb{Z}/\langle 12\rangle$.

Now, suppose G is nonabelian. We would like to divide G by its "non-abelian parts," making them zero and leaving only "abelian parts" in the resulting quotient.

A commutator is an element of the form $aba^{-1}b^{-1}$. Since G is nonabelian, *there are non-identity commutators:* $aba^{-1}b^{-1} \neq e$ *in* G.

$ab = ba$ 

$ab \neq ba$ 

In this case, the set $C := \{aba^{-1}b^{-1} \mid a, b \in G\}$ contains *more* than the identity.

Define the commutator subgroup G' of G to be

$$G' := \langle aba^{-1}b^{-1} \mid a, b \in G\rangle .$$

This is a normal subgroup of G (homework exercise). If we quotient out by it, we get an abelian group! (Because we have killed every instance of the "$ab \neq ba$ pattern" shown above.)

Commutator subgroups and abelianizations

Definition

The abelianization of G is the quotient group G/G'. This is the group that one gets by "killing off" all nonabelian parts of G.

In some sense, the commutator subgroup G' is the smallest normal subgroup N of G such that G/N is abelian. [Note that G would be the "largest" such subgroup.]

Equivalently, the quotient G/G' is the largest abelian quotient of G. [Note that $G/G \cong \langle e \rangle$ would be the "smallest" such quotient.]

Universal property of commutator subgroups

Suppose $f\colon G \to A$ is a homomorphism to an abelian group A. Then there is a unique homomorphism $h\colon G/G' \to A$ such that $f = hq$:

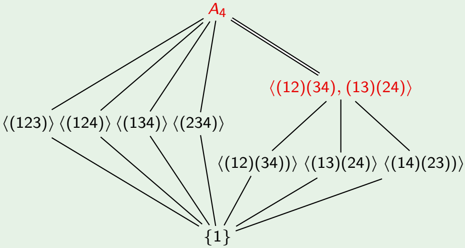
$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "q"'] & & A \\ & G/G' \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

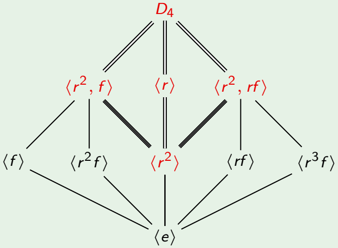
We say that f "factors through" the abelianization, G/G'.

Examples

Consider the groups A_4 and D_4. It is easy to check that

$$G'_{A_4} = \langle xyx^{-1}y^{-1} \mid x, y \in A_4 \rangle \cong V_4 , \qquad G'_{D_4} = \langle xyx^{-1}y^{-1} \mid x, y \in D_4 \rangle = \langle r^2 \rangle .$$





By the *Correspondence Theorem*, the abelianization of A_4 is $A_4/V_4 \cong C_3$, and the abelianization of D_4 is $D_4/\langle r^2\rangle \cong V_4$.

Notice that G/G' is abelian, and moreover, taking the quotient of G by *anything above* G' will yield an abelian group.