Lecture 4.6: Automorphisms

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 4120, Modern Algebra

Basic concepts

Definition

An automorphism is an isomorphism from a group to itself.

The set of all automorphisms of G forms a group, called the automorphism group of G, and denoted $\mathrm{Aut}(G)$.

Remarks.

- An automorphism is determined by where it sends the generators.
- An automorphism ϕ must send generators to generators. In particular, if G is cyclic, then it determines a permutation of the set of (all possible) generators.

Examples

1. There are two automorphisms of $\mathbb{Z}$: the identity, and the mapping $n \mapsto -n$. Thus, $\mathrm{Aut}(\mathbb{Z}) \cong C_2$.
2. There is an automorphism $\phi \colon \mathbb{Z}_5 \to \mathbb{Z}_5$ for each choice of $\phi(1) \in \{1, 2, 3, 4\}$. Thus, $\mathrm{Aut}(\mathbb{Z}_5) \cong C_4$ or V_4. (Which one?)
3. An automorphism ϕ of $V_4 = \langle h, v \rangle$ is determined by the image of h and v. There are 3 choices for $\phi(h)$, and then 2 choices for $\phi(v)$. Thus, $|\mathrm{Aut}(V_4)| = 6$, so it is either $C_6 \cong C_2 \times C_3$, or S_3. (Which one?)

Automorphism groups of $\mathbb{Z}_n$

Definition

The multiplicative group of integers modulo n, denoted $\mathbb{Z}_n^*$ or $U(n)$, is the group

$$U(n) := \{k \in \mathbb{Z}_n \mid \gcd(n,k) = 1\}$$

where the binary operation is multiplication, modulo n.

$U(5) = \{1,2,3,4\} \cong C_4$

	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$U(6) = \{1,5\} \cong C_2$

	1	5
1	1	5
5	5	1

$U(8) = \{1,3,5,7\} \cong C_2 \times C_2$

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Proposition (homework)

The automorphism group of $\mathbb{Z}_n$ is $\mathrm{Aut}(\mathbb{Z}_n) = \{\sigma_a \mid a \in U(n)\} \cong U(n)$, where

$$\sigma_a \colon \mathbb{Z}_n \longrightarrow \mathbb{Z}_n\,, \qquad \sigma_a(1) = a\,.$$

Automorphisms of D_3

Let's find all automorphisms of $D_3 = \langle r, f \rangle$. We'll see a very similar example to this when we study Galois theory.

Clearly, every automorphism ϕ is completely determined by $\phi(r)$ and $\phi(f)$.

Since automorphisms preserve order, if $\phi \in \mathrm{Aut}(D_3)$, then

$$\phi(e) = e\,, \qquad \phi(r) = \underbrace{r \text{ or } r^2}_{\text{2 choices}}, \qquad \phi(f) = \underbrace{f,\ rf, \text{ or } r^2 f}_{\text{3 choices}}\,.$$

Thus, there are *at most* $2 \cdot 3 = 6$ automorphisms of D_3.

Let's try to define two maps, (i) $\alpha \colon D_3 \to D_3$ fixing r, and (ii) $\beta \colon D_3 \to D_3$ fixing f:

$$\begin{cases} \alpha(r) = r \\ \alpha(f) = rf \end{cases} \qquad\qquad \begin{cases} \beta(r) = r^2 \\ \beta(f) = f \end{cases}$$

I claim that:

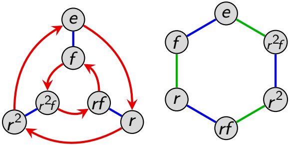
- these both define automorphisms (check this!)
- these generate six *different* automorphisms, and thus $\langle \alpha, \beta \rangle \cong \mathrm{Aut}(D_3)$.

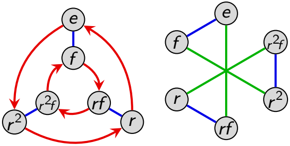
To determine what group this is isomorphic to, find these six automorphisms, and make a group presentation and/or multiplication table. Is it abelian?

Automorphisms of D_3

An automorphism can be thought of as a re-wiring of the Cayley diagram.

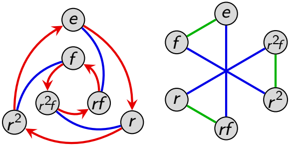
$r \xmapsto{id} r$
$f \longmapsto f$

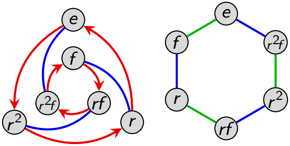




$r \xmapsto{\beta} r^2$
$f \longmapsto f$

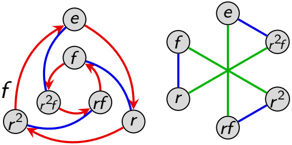
$r \xmapsto{\alpha} r$
$f \longmapsto rf$

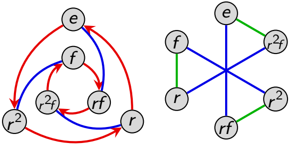




$r \xmapsto{\alpha\beta} r^2$
$f \longmapsto r^2 f$

$r \xmapsto{\alpha^2} r$
$f \longmapsto r^2 f$



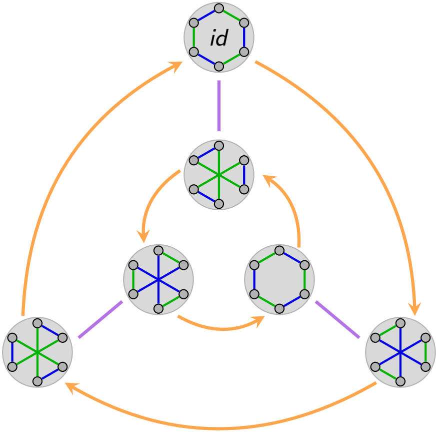


$r \xmapsto{\alpha^2\beta} r^2$
$f \longmapsto rf$

Automorphisms of D_3

Here is the multiplication table and Cayley diagram of $\text{Aut}(D_3) = \langle \alpha, \beta \rangle$.

	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
id	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
α	α	α^2	id	$\alpha\beta$	$\alpha^2\beta$	β
α^2	α^2	id	α	$\alpha^2\beta$	β	$\alpha\beta$
β	β	$\alpha^2\beta$	$\alpha\beta$	id	α^2	α
$\alpha\beta$	$\alpha\beta$	β	$\alpha^2\beta$	α	id	α^2
$\alpha^2\beta$	$\alpha^2\beta$	$\alpha\beta$	β	α^2	α	id



It is purely coincidence that $\text{Aut}(D_3) \cong D_3$. For example, we've already seen that

$$\text{Aut}(\mathbb{Z}_5) \cong U(5) \cong C_4\,, \qquad \text{Aut}(\mathbb{Z}_6) \cong U(6) \cong C_2\,, \qquad \text{Aut}(\mathbb{Z}_8) \cong U(8) \cong C_2 \times C_2\,.$$

Automorphisms of $V_4 = \langle h, v \rangle$

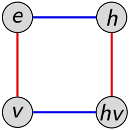
The following permutations are both automorphisms:

α : 

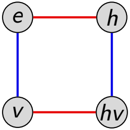
and

β : 

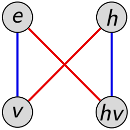
$$h \overset{id}{\longmapsto} h \qquad v \longmapsto v \qquad hv \longmapsto hv$$



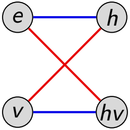
$$h \overset{\beta}{\longmapsto} v \qquad v \longmapsto h \qquad hv \longmapsto hv$$



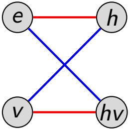
$$h \overset{\alpha}{\longmapsto} v \qquad v \longmapsto hv \qquad hv \longmapsto h$$



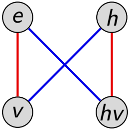
$$h \overset{\alpha\beta}{\longmapsto} h \qquad v \longmapsto hv \qquad hv \longmapsto v$$



$$h \overset{\alpha^2}{\longmapsto} hv \qquad v \longmapsto h \qquad hv \longmapsto v$$



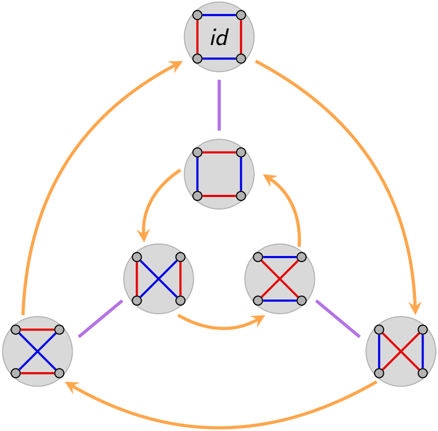
$$h \overset{\alpha^2\beta}{\longmapsto} hv \qquad v \longmapsto v \qquad hv \longmapsto h$$



Automorphisms of $V_4 = \langle h, v \rangle$

Here is the multiplication table and Cayley diagram of $\text{Aut}(V_4) = \langle \alpha, \beta \rangle \cong S_3 \cong D_3$.

	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
id	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
α	α	α^2	id	$\alpha\beta$	$\alpha^2\beta$	β
α^2	α^2	id	α	$\alpha^2\beta$	β	$\alpha\beta$
β	β	$\alpha^2\beta$	$\alpha\beta$	id	α^2	α
$\alpha\beta$	$\alpha\beta$	β	$\alpha^2\beta$	α	id	α^2
$\alpha^2\beta$	$\alpha^2\beta$	$\alpha\beta$	β	α^2	α	id



Recall that α and β can be thought of as the permutations 

and 

and so $\text{Aut}(G) \hookrightarrow \text{Perm}(G) \cong S_n$ always holds.